GKZ hypergeometric systems of the four-loop vacuum Feynman integrals

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Abstract

Basing on Mellin-Barnes representations and Miller's transformation, we present the Gel'fand-Kapranov-Zelevinsky (GKZ) hypergeometric systems of the four-loop vacuum Feynman integrals with arbitrary masses. Through the GKZ hypergeometric systems, the analytical hypergeometric series solutions of the four-loop vacuum Feynman integrals with arbitrary masses can be obtained in neighborhoods of origin including infinity. The analytical expressions of the four-loop vacuum Feynman integrals can be formulated as a linear combination of the corresponding fundamental solution systems in certain convergent region.

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Contents

I. INTRODUCTION

With the improvement of experimental measurement accuracy at the planned future colliders [1–6], Feynman integrals need to be calculated beyond two-loop order. Vacuum integrals are the important subsets of Feynman integrals, which constitute a main building block in asymptotic expansions of Feynman integrals [7, 8] and are also useful for massless theories when a propagator mass is introduced as an intermediate infrared regulator [9]. The calculation of multi-loop vacuum integrals is a good breakthrough window in the calculation of multi-loop Feynman integrals. In this article, we investigate the analytical calculation for the four-loop vacuum integrals with arbitrary masses.

It's well known that the completely general one-loop integrals are analytically in the time-space dimension $D = 4 - 2\varepsilon$ [10–13]. At the two-loop level, the vacuum integrals have been calculated to polylogarithms or equivalent functions [14–18]. The three-loop vacuum integrals are also calculated analytically and numerically in some literatures [19–40]. But, very few four-loop vacuum integrals are calculated analytically. The vacuum integrals at the four-loop level are only calculated analytically for single-mass-scale [41, 42], equal masses [43], and reduction [44, 45]. Recently, Feynman integrals using auxiliary mass flow numerical method [46, 47], also can be reduced to vacuum integrals, which can be numerical solved by further reduction. In order to further improve the computational efficiency and give analytical results completely, it is meaningful to explore new analytical calculating method of the four-loop vacuum integrals with arbitrary masses.

During the past decades, Feynman integrals have been considered as the generalized hypergeometric functions [48–72]. Considering Feynman integrals as the generalized hypergeometric functions, one finds that the $D-$module of a Feynman diagram [64, 73] is isomorphic to Gel'fand-Kapranov-Zelevinsky (GKZ) $D-$module [74–78]. GKZ-hypergeometric systems of Feynman integrals with codimension$= 0,\ 1$ are presented in Refs. [79, 80] through Lee-Pomeransky parametric representations [81]. To construct canonical series solutions with suitable independent variables, one should compute the restricted D-module of GKZ-hypergeometric system originating from Lee-Pomeransky representations on corresponding hyperplane in the parameter space [82–84]. In our previous work, GKZ hypergeometric systems of one- and two-loop Feynman diagrams are also obtained [85–87] from Mellin-Barnes representations [68, 69], through Miller's transformation [88, 89]. There are some recent

work in GKZ framework of Feynman integrals [90–108].

In our previous work, we have given GKZ hypergeometric systems of the Feynman integrals of the two-loop vacuum integral [85] and three-loop vacuum integrals [40]. In this article, we derive GKZ hypergeometric systems of the four-loop vacuum integrals with arbitrary masses, basing on Mellin-Barnes representations and Miller's transformation. The generally strategy for analyzing the four-loop vacuum integrals includes three steps here. Firstly, we obtain the Mellin-Barnes representation of the vacuum integral. Secondly, we find GKZ hypergeometric system of the vacuum integral via Miller's transformation. Finally, analytical hypergeometric series solutions of the vacuum integral are constructed in neighborhoods of origin including infinity. The integration constants, i.e. the combination coefficients, are determined from the vacuum integral of an ordinary point or some regular singularities.

Our presentation is organized as following. Through the Mellin-Barnes representation and Miller's transformation, we derive the GKZ hypergeometric system of the four-loop vacuum integral with five propagates in Sec. II, six propagates in Sec. III, and seven propagates in Sec. IV. And then, we construct the analytical hypergeometric series solutions of the GKZ system of the four-loop vacuum integrals in Sec. V. At last, the conclusions are summarized in Sec. VI, and some formulates are presented in the appendices.

II. GKZ HYPERGEOMETRIC SYSTEM OF THE FOUR-LOOP VACUUM INTEGRAL WITH FIVE PROPAGATES

The general analytic expression for the Feynman integral of the four-loop vacuum diagram with five propagates in Fig. 1 is written as

$$U_5 = \left(\Lambda^2_{\rm RE}\right)^{8-2D} \int \frac{d^D\mathbf{q}}{(2\pi)^D} \frac{1}{(q_1^2-m_1^2)(q_2^2-m_2^2)(q_3^2-m_3^2)} \\ \times \frac{1}{[(q_1+q_2+q_3+q_4)^2-m_4^2](q_4^2-m_5^2)}\;, \tag{1}$$

where $D=4-2\varepsilon$ is the number of dimensions in dimensional regularization and $\Lambda_{\rm RE}$ denotes the renormalization energy scale, $\mathbf{q}=(q_1,\;\cdots,\;q_4)$, $\frac{d^D\mathbf{q}}{(2\pi)^D} = \frac{d^Dq_1}{(2\pi)^D}\frac{d^Dq_2}{(2\pi)^D}\frac{d^Dq_3}{(2\pi)^D}\frac{d^Dq_4}{(2\pi)^D}$. The Feynman integral is hard to calculate analytically, if all virtual masses are nonzero. So, one can extract the virtual masses from the integral, to facilitate further calculation. Adopting

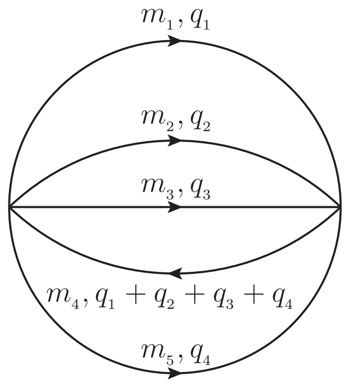

FIG. 1: Four-loop vacuum diagram with five propagators, which m_i denotes the mass of the i-th particle and q_j denotes the momentum.

the notation of Refs. [68, 69], the Feynman integral of the four-loop vacuum diagram with five propagates can be written as

$$U_5 = \frac{\left(\Lambda^2_{\rm RE}\right)^{8-2D}}{(2\pi i)^4} \int_{-i\infty}^{+i\infty} d\mathbf{s} \Big[\prod_{i=1}^{4} (-m_i^2)^{s_i} \Gamma(-s_i)\Gamma(1+s_i)\Big] I_q \, , \tag{2}$$

where $\mathbf{s} = (s_1, \cdots, s_4)$, $d\mathbf{s} = ds_1 ds_2 ds_3 ds_4$ and

$$I_q \equiv \int \frac{d^D \mathbf{q}}{(2\pi)^D} \frac{1}{(q_1^2)^{1+s_1} (q_2^2)^{1+s_2} (q_3^2)^{1+s_3} [(q_1 + q_2 + q_3 + q_4)^2]^{1+s_4} (q_4^2 - m_5^2)} \, . \tag{3}$$

The integral I_q just keeps one mass m_5, which can be easily calculated analytically. The calculation of the integral I_q can be seen in Appendix A.

And then, the Mellin-Barnes representation of the Feynman integral of the four-loop vacuum diagram in Eq. (2) can be written as

$$\begin{aligned} U_5 = & \frac{-m_5^6}{(2\pi i)^4 (4\pi)^8} \Big(\frac{4\pi \Lambda^2_{\rm RE}}{m_5^2}\Big)^{8-2D} \int_{-i\infty}^{+i\infty} d\mathbf{s} \Big[\prod_{i=1}^{4} \Big(\frac{m_i^2}{m_5^2}\Big)^{s_i} \Gamma(-s_i)\Big] \\ & \times \Big[\prod_{i=1}^{4} \Gamma(\frac{D}{2} - 1 - s_i)\Big] \Gamma(4 - \frac{3D}{2} + \sum_{i=1}^{4} s_i) \Gamma(5 - 2D + \sum_{i=1}^{4} s_i) \, . \end{aligned} \tag{4}$$

It is well known that negative integers and zero are simple poles of the function $\Gamma(z)$. As all s_i contours are closed to the right in corresponding complex planes, one finds that the analytic expression of the Feynman integral can be written as the linear combination of generalized hypergeometric functions. Taking the residue of the pole of $\Gamma(-s_i)$, $(i =$

$1,\cdots,\ 4)$, we can derive one linear independent term of the integral:

$$U_{_5} \ni \frac{-m_{_5}^6}{(4\pi)^8}\Big(\frac{4\pi\Lambda_{_{\rm RE}}^2}{m_{_5}^2}\Big)^{8-2D}\sum_{n_1=0}^\infty\sum_{n_2=0}^\infty\sum_{n_3=0}^\infty\sum_{n_4=0}^\infty(-)^{\sum\limits_{i=1}^4 n_i}x_{_1}^{n_1}x_{_2}^{n_2}x_{_3}^{n_3}x_{_4}^{n_4}$$
$$\times\Big[\prod_{i=1}^4\Gamma(\frac{D}{2}-1-n_{_i})(n_{_i}!)^{-1}\Big]\Gamma(4-\frac{3D}{2}+\sum_{i=1}^4 n_{_i})\Gamma(5-2D+\sum_{i=1}^4 n_{_i})\;, \tag{5}$$

with $x_{_i}=\frac{m_{_i}^2}{m_{_5}^2}$, $(i=1,\cdots,\ 4)$.

We adopt the identity

$$\Gamma(z-n)\Gamma(1-z+n)=(-)^n\Gamma(z)\Gamma(1-z)=(-)^n\pi/\sin\pi z\;, \tag{6}$$

which originate from the well-known relation $\Gamma(z)\Gamma(1-z)=\pi/\sin\pi z$. Then, Eq. (5) can be written as

$$U_{_5}\ni\frac{-m_{_5}^6}{(4\pi)^8}\Big(\frac{4\pi\Lambda_{_{\rm RE}}^2}{m_{_5}^2}\Big)^{8-2D}\frac{\pi^4}{\sin^4\frac{\pi D}{2}}T_{_5}(\mathbf{a},\ \mathbf{b}\;\Big|\;\mathbf{x})\;, \tag{7}$$

with

$$T_{_5}(\mathbf{a},\ \mathbf{b}\;\Big|\;\mathbf{x})=\sum_{n_1=0}^\infty\sum_{n_2=0}^\infty\sum_{n_3=0}^\infty\sum_{n_4=0}^\infty A_{_{n_1n_2n_3n_4}}x_{_1}^{n_1}x_{_2}^{n_2}x_{_3}^{n_3}x_{_4}^{n_4}\;, \tag{8}$$

where $\mathbf{x}=(x_{_1},x_{_2},x_{_3},x_{_4})$, $\mathbf{a}=(a_{_1},a_{_2})$ and $\mathbf{b}=(b_{_1},b_{_2},b_{_3},b_{_4})$ with

$$a_{_1}=4-\frac{3D}{2},\quad a_{_2}=5-2D,\quad b_{_1}=b_{_2}=b_{_3}=b_{_4}=2-\frac{D}{2}\;, \tag{9}$$

and the coefficient $A_{_{n_1n_2n_3n_4}}$ is

$$A_{_{n_1n_2n_3n_4}}=\frac{\Gamma(a_{_1}+\sum\limits_{i=1}^4 n_{_i})\Gamma(a_{_2}+\sum\limits_{i=1}^4 n_{_i})}{\prod\limits_{i=1}^4 n_{_i}!\Gamma(b_{_i}+n_{_i})}\;. \tag{10}$$

Here, we just derive one linear independent term of the integral. We still need to derive other linear independent terms of the integral.

Through the adjacent relations of the coefficient $A_{_{n_1n_2n_3n_4}}$:

$$\frac{A_{_{(n_1+1)n_2n_3n_4}}}{A_{_{n_1n_2n_3n_4}}}=\frac{(a_{_1}+\sum\limits_{i=1}^4 n_{_i})(a_{_2}+\sum\limits_{i=1}^4 n_{_i})}{(n_{_1}+1)(b_{_1}+n_{_1})}\;,$$
$$\frac{A_{_{n_1(n_2+1)n_3n_4}}}{A_{_{n_1n_2n_3n_4}}}=\frac{(a_{_1}+\sum\limits_{i=1}^4 n_{_i})(a_{_2}+\sum\limits_{i=1}^4 n_{_i})}{(n_{_2}+1)(b_{_2}+n_{_2})}\;,$$

$$
\begin{aligned}
\frac{A_{n_1 n_2 (n_3+1) n_4}}{A_{n_1 n_2 n_3 n_4}} &= \frac{(a_1 + \sum_{i=1}^{4} n_i)(a_2 + \sum_{i=1}^{4} n_i)}{(n_3+1)(b_3+n_3)} \;, \\
\frac{A_{n_1 n_2 n_3 (n_4+1)}}{A_{n_1 n_2 n_3 n_4}} &= \frac{(a_1 + \sum_{i=1}^{4} n_i)(a_2 + \sum_{i=1}^{4} n_i)}{(n_4+1)(b_4+n_4)} \;,
\end{aligned}
\tag{11}
$$

the difference-differential operators are written as

$$
\begin{aligned}
&(\sum_{i=1}^{4} \vartheta_{x_i} + a_j) T_5(\mathbf{a},\ \mathbf{b} \,\big|\, \mathbf{x}) = a_j T_5(\mathbf{a} + \mathbf{e}_{2,j},\ \mathbf{b} \,\big|\, \mathbf{x}) \;,\ (j = 1, 2) \;, \\
&(\vartheta_{x_k} + b_k - 1) T_5(\mathbf{a},\ \mathbf{b} \,\big|\, \mathbf{x}) = (b_k - 1) T_5(\mathbf{a},\ \mathbf{b} - \mathbf{e}_{4,k} \,\big|\, \mathbf{x}) \;, \\
&\partial_{x_k} T_5(\mathbf{a},\ \mathbf{b} \,\big|\, \mathbf{x}) = \frac{a_1 a_2}{b_k} T_5(\mathbf{a} + \mathbf{e}_2,\ \mathbf{b} + \mathbf{e}_{4,k} \,\big|\, \mathbf{x}) \;,\ (k = 1, \cdots, 4) \;.
\end{aligned}
\tag{12}
$$

Here $\mathbf{e}_{2,j} \in \mathbf{R}^2$ $(j = 1, 2)$ with $\mathbf{e}_{2,1} = (1,\ 0)$ and $\mathbf{e}_{2,2} = (0,\ 1)$, $\mathbf{e}_2 = (1,\ 1)$, $\mathbf{e}_{4,k} \in \mathbf{R}^4$ $(k = 1, \cdots, 4)$ denotes the row vector whose entry is zero except that the $k-$th entry is 1, $\mathbf{e}_4 = (1,\ 1,\ 1,\ 1)$, $\vartheta_{x_k} = x_k \partial_{x_k}$ denotes the Euler operators, and $\partial_{x_k} = \partial / \partial x_k$, respectively.

We can define the auxiliary function

$$
\Phi_5(\mathbf{a},\ \mathbf{b} \,\big|\, \mathbf{x},\ \mathbf{u},\ \mathbf{v}) = \mathbf{u}^{\mathbf{a}} \mathbf{v}^{\mathbf{b} - \mathbf{e}_4} T_5(\mathbf{a},\ \mathbf{b} \,\big|\, \mathbf{x}) \;,
\tag{13}
$$

with the intermediate variables $\mathbf{u} = (u_1,\ u_2) = (1,\ 1)$, $\mathbf{v} = (v_1,\ v_2,\ v_3,\ v_4) = (1,\ 1,\ 1,\ 1)$. Through Miller's transformation [88, 89], the relations are obtained

$$
\begin{aligned}
&\vartheta_{u_j} \Phi_5(\mathbf{a},\ \mathbf{b} \,\big|\, \mathbf{x},\ \mathbf{u},\ \mathbf{v}) = a_j \Phi_5(\mathbf{a},\ \mathbf{b} \,\big|\, \mathbf{x},\ \mathbf{u},\ \mathbf{v}) \;,\ (j = 1, 2) \;, \\
&\vartheta_{v_k} \Phi_5(\mathbf{a},\ \mathbf{b} \,\big|\, \mathbf{x},\ \mathbf{u},\ \mathbf{v}) = (b_k - 1) \Phi_5(\mathbf{a},\ \mathbf{b} \,\big|\, \mathbf{x},\ \mathbf{u},\ \mathbf{v}) \;,\ (k = 1, \cdots, 4) \;,
\end{aligned}
\tag{14}
$$

which naturally induces the notion of GKZ hypergeometric system.

In addition, the contiguous relations of Eq. (12) are rewritten as

$$
\begin{aligned}
&u_j (\sum_{i=1}^{4} \vartheta_{x_i} + \vartheta_{u_j}) \Phi_5(\mathbf{a},\ \mathbf{b} \,\big|\, \mathbf{x},\ \mathbf{u},\ \mathbf{v}) = a_j \Phi_5(\mathbf{a} + \mathbf{e}_{2,j},\ \mathbf{b} \,\big|\, \mathbf{x},\ \mathbf{u},\ \mathbf{v}), \\
&\frac{1}{v_k} (\vartheta_{x_k} + \vartheta_{v_k}) \Phi_5(\mathbf{a},\ \mathbf{b} \,\big|\, \mathbf{x},\ \mathbf{u},\ \mathbf{v}) = (b_k - 1) \Phi_5(\mathbf{a},\ \mathbf{b} - \mathbf{e}_{4,k} \,\big|\, \mathbf{x},\ \mathbf{u},\ \mathbf{v}), \\
&u_1 u_2 v_k \partial_{x_k} \Phi_5(\mathbf{a},\ \mathbf{b} \,\big|\, \mathbf{x},\ \mathbf{u},\ \mathbf{v}) = \frac{a_1 a_2}{b_k} \Phi_5(\mathbf{a} + \mathbf{e}_2,\ \mathbf{b} + \mathbf{e}_{4,k} \,\big|\, \mathbf{x},\ \mathbf{u},\ \mathbf{v}) \;.
\end{aligned}
\tag{15}
$$

The operators above together with ϑ_{u_j}, ϑ_{v_j} define the Lie algebra of the hypergeometric systems [88, 89]. Through the transformation

$$
z_j = \frac{1}{u_j},\ (j = 1, 2),\quad z_{2+k} = v_k,\ z_{6+k} = \frac{x_k}{u_1 u_2 v_k},\ (k = 1, \cdots, 4),
\tag{16}
$$

one derives

$$
\begin{aligned}
\vartheta_{u_1} &= -\vartheta_{z_1} - \vartheta_{z_7} - \vartheta_{z_8} - \vartheta_{z_9} - \vartheta_{z_{10}} \;, \\
\vartheta_{u_2} &= -\vartheta_{z_2} - \vartheta_{z_7} - \vartheta_{z_8} - \vartheta_{z_9} - \vartheta_{z_{10}} \,, \\
\vartheta_{v_k} &= \vartheta_{z_{2+k}} - \vartheta_{z_{6+k}} \,, \quad \vartheta_{x_k} = \vartheta_{z_{6+k}} \;.
\end{aligned}
\tag{17}
$$

Through Eq. (14), one can finally have the GKZ hypergeometric system for the four-loop vacuum integral with five propagates:

$$
\mathbf{A_5} \cdot \vec{\vartheta}_5 \Phi_5 = \mathbf{B_5} \Phi_5 \;, \tag{18}
$$

where

$$
\begin{aligned}
\mathbf{A_5} &= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1
\end{pmatrix} , \\
\vec{\vartheta}_5^{\,T} &= (\vartheta_{z_1} ,\ \cdots ,\ \vartheta_{z_{10}}) \;, \\
\mathbf{B_5}^T &= (-a_1 ,\ -a_2 ,\ b_1 - 1 ,\ b_2 - 1 ,\ b_3 - 1 ,\ b_4 - 1) \;.
\end{aligned}
\tag{19}
$$

Correspondingly the dual matrix $\tilde{\mathbf{A}}_5$ of $\mathbf{A_5}$ is

$$
\tilde{\mathbf{A}}_5 = \begin{pmatrix}
-1 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
-1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
-1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{pmatrix} . \tag{20}
$$

The row vectors of the matrix $\tilde{\mathbf{A}}_5$ induce the integer sublattice $\mathbf{B}$ which can be used to construct the formal solutions in hypergeometric series.

Defining the combined variables

$$
y_1 = \frac{z_3 z_7}{z_1 z_2} \;, \quad y_2 = \frac{z_4 z_8}{z_1 z_2} \;, \quad y_3 = \frac{z_5 z_9}{z_1 z_2} \;, \quad y_4 = \frac{z_6 z_{10}}{z_1 z_2} \;, \tag{21}
$$

we write the solutions satisfying Eq. (18) as

$$
\Phi_5(\mathbf{z}) = \Big(\prod_{i=1}^{10} z_i^{\alpha_i} \Big) \varphi_5(y_1 ,\ y_2 ,\ y_3 ,\ y_4) \;. \tag{22}
$$

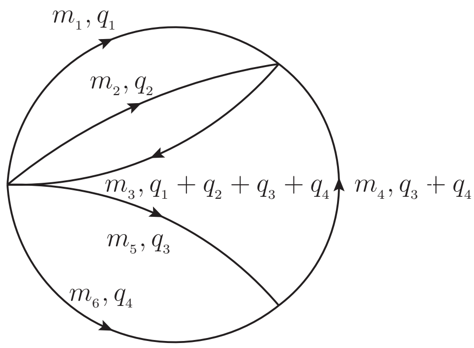

FIG. 2: Four-loop vacuum diagram with six propagators for type A, which m_i denotes the mass of the i-th particle and q_j denotes the momentum.

Here $\vec{\alpha}^T = (\alpha_1, \cdots, \alpha_{10})$ denotes a sequence of complex number such that

$$\mathbf{A_5} \cdot \vec{\alpha} = \mathbf{B_5} \ , \tag{23}$$

namely,

$$\begin{aligned}
&\alpha_1 + \alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10} = -a_1 \ , \quad \alpha_2 + \alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10} = -a_2 \ , \\
&\alpha_3 - \alpha_7 = b_1 - 1 \ , \quad \alpha_4 - \alpha_8 = b_2 - 1 \ , \quad \alpha_5 - \alpha_9 = b_3 - 1 \ , \quad \alpha_6 - \alpha_{10} = b_4 - 1 \ .
\end{aligned} \tag{24}$$

In the following Sec. V, we will show the analytical hypergeometric series solutions solving from the GKZ hypergeometric system in Eq. (18).

III. GKZ HYPERGEOMETRIC SYSTEM OF THE FOUR-LOOP VACUUM INTEGRALS WITH SIX PROPAGATES

A. Four-loop vacuum diagram with six propagates for type A

The four-loop vacuum diagrams with six propagates have two topologies, which one can be seen in Fig. 2. The general analytic expression for the Feynman integral of the four-loop vacuum diagram with six propagates for type A in Fig. 2 is written as

$$\begin{aligned}
U_{6A} = \left(\Lambda^2_{\rm RE}\right)^{8-2D} \int \frac{d^D\mathbf{q}}{(2\pi)^D} & \frac{1}{(q_1^2 - m_1^2)(q_2^2 - m_2^2)[(q_1 + q_2 + q_3 + q_4)^2 - m_3^2]} \\
\times & \frac{1}{[(q_3 + q_4)^2 - m_4^2](q_3^2 - m_5^2)(q_4^2 - m_6^2)} .
\end{aligned} \tag{25}$$

Through the Mellin-Barnes transformation, the Feynman integral can be written as

$$U_{_{6A}}=\frac{\Big(\Lambda_{_{\rm RE}}^2\Big)^{8-2D}}{(2\pi i)^5}\int_{-i\infty}^{+i\infty}d{\bf s}\Big[\prod_{i=1}^{5}(-m_{_i}^2)^{s_i}\Gamma(-s_{_i})\Gamma(1+s_{_i})\Big]I_{_{6A}}\;, \tag{26}$$

where ${\bf s}=(s_{_1},\;\cdots,\;s_{_5})$, and

$$I_{_{6A}}\equiv\int\frac{d^D{\bf q}}{(2\pi)^D}\frac{1}{(q_{_1}^2)^{1+s_1}(q_{_2}^2)^{1+s_2}[(q_{_1}+q_{_2}+q_{_3}+q_{_4})^2]^{1+s_3}}$$
$$\times\frac{1}{[(q_{_3}+q_{_4})^2]^{1+s_4}(q_{_3}^2)^{1+s_5}(q_{_4}^2-m_{_6}^2)}. \tag{27}$$

Similarly, integrating out ${\bf q}$, the Mellin-Barnes representation of the Feynman integral of the four-loop vacuum diagram with six propagates for type A can be written as

$$U_{_{6A}}=\frac{m_{_6}^4}{(2\pi i)^5(4\pi)^8}\Big(\frac{4\pi\Lambda_{_{\rm RE}}^2}{m_{_6}^2}\Big)^{8-2D}\int_{-i\infty}^{+i\infty}d{\bf s}\Big[\prod_{i=1}^{5}\Big(\frac{m_{_i}^2}{m_{_6}^2}\Big)^{s_i}\Gamma(-s_{_i})\Big]$$
$$\times\Big[\prod_{i=1}^{3}\Gamma(\frac{D}{2}-1-s_{_i})\Big]\Gamma(\frac{D}{2}-1-s_{_5})\Gamma(5-\frac{3D}{2}+\sum_{i=1}^{5}s_{_i})$$
$$\times\frac{\Gamma(6-2D+\sum\limits_{i=1}^{5}s_{_i})\Gamma(3-D+\sum\limits_{i=1}^{3}s_{_i})\Gamma(\frac{3D}{2}-4-\sum\limits_{i=1}^{4}s_{_i})\Gamma(1+s_{_4})}{\Gamma(4-D+\sum\limits_{i=1}^{4}s_{_i})\Gamma(\frac{3D}{2}-3-\sum\limits_{i=1}^{3}s_{_i})}\;. \tag{28}$$

Taking the residue of the pole of $\Gamma(-s_{_i})$, $(i=1,\cdots,5)$, one can derive one linear independent term of the vacuum integral:

$$U_{_{6A}}\ni\frac{-m_{_6}^4}{(4\pi)^8}\Big(\frac{4\pi\Lambda_{_{\rm RE}}^2}{m_{_6}^2}\Big)^{8-2D}\sum_{\bf n=0}^{\infty}{\bf x}^{\bf n}(-)^{\sum\limits_{i=1}^{5}n_i}\Big[\prod_{i=1}^{3}\Gamma(\frac{D}{2}-1-n_{_i})\Big]$$
$$\times\Gamma(\frac{D}{2}-1-n_{_5})\Gamma(5-\frac{3D}{2}+\sum_{i=1}^{5}n_{_i})\Gamma(6-2D+\sum_{i=1}^{5}n_{_i})$$
$$\times\frac{\Gamma(3-D+\sum\limits_{i=1}^{3}n_{_i})\Gamma(\frac{3D}{2}-4-\sum\limits_{i=1}^{4}n_{_i})\Gamma(1+n_{_4})}{\Big[\prod\limits_{i=1}^{5}n_{_i}!\Big]\Gamma(4-D+\sum\limits_{i=1}^{4}n_{_i})\Gamma(\frac{3D}{2}-3-\sum\limits_{i=1}^{3}n_{_i})}, \tag{29}$$

with ${\bf n}=(n_{_1},\;\cdots,\;n_{_5})$, ${\bf x}=(x_{_1},\;\cdots,\;x_{_5})$, $x_{_i}=\frac{m_{_i}^2}{m_{_6}^2}$, $(i=1,\cdots,5)$. Adopting the identity in Eq. (6), Eq. (29) can be written as

$$U_{_{6A}}\ni\frac{m_{_6}^4}{(4\pi)^8}\Big(\frac{4\pi\Lambda_{_{\rm RE}}^2}{m_{_6}^2}\Big)^{8-2D}\frac{\pi^4}{\sin^4\frac{\pi D}{2}}T_{_{6A}}({\bf a},\;{\bf b}\;\Big|\;{\bf x})\;, \tag{30}$$

with

$$T_{_{6A}}({\bf a},\;{\bf b}\;\Big|\;{\bf x})=\sum_{\bf n=0}^{\infty}A_{_{\bf n}}{\bf x}^{\bf n}, \tag{31}$$

where $\mathbf{a}=(a_1,\ \cdots,a_5)$ and $\mathbf{b}=(b_1,\ \cdots,b_6)$ with

$$
\begin{aligned}
& a_1=5-\frac{3D}{2},\ a_2=6-2D,\ a_3=3-D,\ a_4=4-\frac{3D}{2},\ a_5=1, \\
& b_1=b_2=b_3=b_4=2-\frac{D}{2},\ b_5=4-D,\ b_6=5-\frac{3D}{2}\ ,
\end{aligned} \tag{32}
$$

and the coefficient $A_{\mathbf{n}}$ is

$$
A_{\mathbf{n}}=\frac{\Gamma(a_1+\sum\limits_{i=1}^{5}n_i)\Gamma(a_2+\sum\limits_{i=1}^{5}n_i)\Gamma(a_3+\sum\limits_{i=1}^{3}n_i)\Gamma(a_4+\sum\limits_{i=1}^{3}n_i)\Gamma(a_5+n_4)}{\Big[\prod\limits_{i=1}^{5}n_i!\Big]\Big[\prod\limits_{i=1}^{3}\Gamma(b_i+n_i)\Big]\Gamma(b_4+n_5)\Gamma(b_5+\sum\limits_{i=1}^{4}n_i)\Gamma(b_6+\sum\limits_{i=1}^{4}n_i)}. \tag{33}
$$

In order to proceed with our analysis, we also define the auxiliary function

$$
\Phi_{6A}(\mathbf{a},\ \mathbf{b}\,\Big|\,\mathbf{x},\ \mathbf{u},\ \mathbf{v})=\mathbf{u}^{\mathbf{a}}\mathbf{v}^{\mathbf{b}-\mathbf{e}_6}T_{6A}(\mathbf{a},\ \mathbf{b}\,\Big|\,\mathbf{x})\ , \tag{34}
$$

with the intermediate variables $\mathbf{u}=(u_1,\cdots,u_5)=(1,\ 1,\ 1,\ 1,\ 1)$, $\mathbf{v}=(v_1,\cdots,v_6)$, $\mathbf{v}=\mathbf{e}_6=(1,\ 1,\ 1,\ 1,\ 1,\ 1)$. Then one can obtain

$$
\begin{aligned}
& \vartheta_{u_j}\Phi_{6A}(\mathbf{a},\ \mathbf{b}\,\Big|\,\mathbf{x},\ \mathbf{u},\ \mathbf{v})=a_j\Phi_{6A}(\mathbf{a},\ \mathbf{b}\,\Big|\,\mathbf{x},\ \mathbf{u},\ \mathbf{v})\ ,\ (j=1,\cdots,5)\ , \\
& \vartheta_{v_k}\Phi_{6A}(\mathbf{a},\ \mathbf{b}\,\Big|\,\mathbf{x},\ \mathbf{u},\ \mathbf{v})=(b_k-1)\Phi_{6A}(\mathbf{a},\ \mathbf{b}\,\Big|\,\mathbf{x},\ \mathbf{u},\ \mathbf{v})\ ,\ (k=1,\cdots,6)\ .
\end{aligned} \tag{35}
$$

Through the transformation

$$
\begin{aligned}
& z_j=\frac{1}{u_j},\ \ (j=1,\cdots,5),\ \ z_{5+k}=v_k,\ \ (k=1,\cdots,6), \\
& z_{12}=\frac{x_1}{u_1u_2u_3u_4v_1v_5v_6},\ \ z_{13}=\frac{x_2}{u_1u_2u_3u_4v_2v_5v_6}, \\
& z_{14}=\frac{x_3}{u_1u_2u_3u_4v_3v_5v_6},\ \ z_{15}=\frac{x_4}{u_1u_2u_5v_5v_6},\ \ z_{16}=\frac{x_5}{u_1u_2v_4}\ ,
\end{aligned} \tag{36}
$$

one derives the GKZ hypergeometric system for the four-loop vacuum diagram with six propagates for type A:

$$
\mathbf{A_{6A}}\cdot\vec{\vartheta}_{6A}\Phi_{6A}=\mathbf{B_{6A}}\Phi_{6A}\ , \tag{37}
$$

where

$$
\mathbf{A_{6A}}=\begin{pmatrix}\mathbf{I_{11\times 11}} & \mathbf{A_{X6A}}\end{pmatrix}\ ,
$$

$$
\mathbf{A_{X6A}}^{T} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} ,
$$

$$
\vec{\vartheta}_{6A}^{T} = (\vartheta_{z_1}, \cdots, \vartheta_{z_{16}}) ,
$$

$$
\mathbf{B_{6A}}^{T} = (-a_1, \ \cdots, \ -a_5, \ b_1 - 1, \ \cdots, \ b_6 - 1) . \tag{38}
$$

Here, $\mathbf{I_{11\times 11}}$ is an 11×11 unit matrix.

Correspondingly the dual matrix $\tilde{\mathbf{A}}_{\mathbf{6A}}$ of $\mathbf{A_{6A}}$ is

$$
\tilde{\mathbf{A}}_{\mathbf{6A}} = \begin{pmatrix} -\mathbf{A_{X6A}}^{T} & \mathbf{I_{5\times 5}} \end{pmatrix} . \tag{39}
$$

Here, $\mathbf{I_{5\times 5}}$ is a 5×5 unit matrix. The row vectors of the matrix $\tilde{\mathbf{A}}_{\mathbf{6A}}$ induce the integer sublattice $\mathbf{B}$ which can be used to construct the formal solutions in hypergeometric series.

Defining the combined variables

$$
\begin{aligned}
& y_1 = \frac{z_6 z_{10} z_{11} z_{12}}{z_1 z_2 z_3 z_4} , \quad y_2 = \frac{z_7 z_{10} z_{11} z_{13}}{z_1 z_2 z_3 z_4} , \\
& y_3 = \frac{z_8 z_{10} z_{11} z_{14}}{z_1 z_2 z_3 z_4} , \quad y_4 = \frac{z_{10} z_{11} z_{15}}{z_1 z_2 z_5} , \quad y_5 = \frac{z_9 z_{16}}{z_1 z_2} ,
\end{aligned} \tag{40}
$$

we write the solutions satisfying Eq. (37) as

$$
\Phi_{6A}(\mathbf{z}) = \Big(\prod_{i=1}^{16} z_i^{\alpha_i} \Big) \varphi_{6A}(\mathbf{y}) , \tag{41}
$$

where $\mathbf{y} = (y_1, \ \cdots, \ y_5)$, $\vec{\alpha}^{\,T} = (\alpha_1, \ \cdots, \ \alpha_{16})$ denotes a sequence of complex number such that

$$
\mathbf{A_{6A}} \cdot \vec{\alpha} = \mathbf{B_{6A}} . \tag{42}
$$

B. Four-loop vacuum diagram with six propagates for type B

The general analytic expression for the Feynman integral of the four-loop vacuum diagram with six propagates for type B in Fig. 3 can be written as

$$
\begin{aligned}
U_{6B} = & \Big(\Lambda_{\rm RE}^2\Big)^{8-2D} \int \frac{d^D\mathbf{q}}{(2\pi)^D} \frac{1}{(q_1^2 - m_1^2)[(q_1 + q_3 + q_4)^2 - m_2^2]} \\
& \times \frac{1}{(q_2^2 - m_3^2)[(q_2 + q_3 + q_4)^2 - m_4^2](q_3^2 - m_5^2)(q_4^2 - m_6^2)} .
\end{aligned} \tag{43}
$$

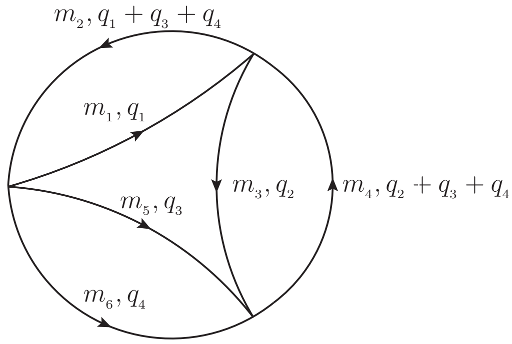

FIG. 3: Four-loop vacuum diagram with six propagators for type B, which m_i denotes the mass of the i-th particle and q_j denotes the momentum.

Integrating out $\mathbf{q}$, the Mellin-Barnes representation of the Feynman integral of the four-loop vacuum diagram with six propagates for type B can be written as

$$U_{6B} = \frac{m_6^4}{(2\pi i)^5(4\pi)^8}\Big(\frac{4\pi\Lambda^2_{\rm RE}}{m_6^2}\Big)^{8-2D}\int_{-i\infty}^{+i\infty} d\mathbf{s}\Big[\prod_{i=1}^{5}\Big(\frac{m_i^2}{m_6^2}\Big)^{s_i}\Gamma(-s_i)\Big]$$
$$\times\Big[\prod_{i=1}^{5}\Gamma(\frac{D}{2}-1-s_i)\Big]\Gamma(5-\frac{3D}{2}+\sum_{i=1}^{5}s_i)\Gamma(6-2D+\sum_{i=1}^{5}s_i)$$
$$\times\frac{\Gamma(\frac{3D}{2}-4-\sum_{i=1}^{4}s_i)\Gamma(2-\frac{D}{2}+s_1+s_2)\Gamma(2-\frac{D}{2}+s_3+s_4)}{\Gamma(4-D+\sum_{i=1}^{4}s_i)\Gamma(D-2-s_1-s_2)\Gamma(D-2-s_3-s_4)}\;, \tag{44}$$

where $\mathbf{s} = (s_1,\ \cdots,\ s_5)$.

Taking the residue of the pole of $\Gamma(-s_i)$, $(i = 1, \cdots, 5)$, one can derive one linear independent term of the integral:

$$U_{6B} \ni \frac{m_6^4}{(4\pi)^8}\Big(\frac{4\pi\Lambda^2_{\rm RE}}{m_6^2}\Big)^{8-2D}\frac{\pi^4\sin^2\pi D}{\sin^5\frac{\pi D}{2}\sin\frac{3\pi D}{2}}T_{6B}(\mathbf{a},\ \mathbf{b}\ \big|\ \mathbf{x})\;, \tag{45}$$

with

$$T_{6B}(\mathbf{a},\ \mathbf{b}\ \big|\ \mathbf{x}) = \sum_{\mathbf{n}=\mathbf{0}}^{\infty}A_{\mathbf{n}}\mathbf{x}^{\mathbf{n}}, \tag{46}$$

where $\mathbf{n} = (n_1,\ \cdots,\ n_5)$, $\mathbf{x} = (x_1,\ \cdots,\ x_5)$, $x_i = \frac{m_i^2}{m_6^2}$, $\mathbf{a} = (a_1,\ \cdots, a_6)$ and $\mathbf{b} = (b_1,\ \cdots, b_7)$ with

$$a_1 = 5-\frac{3D}{2},\ a_2 = 6-2D,\ a_3 = a_5 = 2-\frac{D}{2},\ a_4 = a_6 = 3-D\;,$$
$$b_1 = b_2 = b_3 = b_4 = b_5 = 2-\frac{D}{2},\ b_6 = 4-D,\ b_7 = 5-\frac{3D}{2}\;, \tag{47}$$

and the coefficient $A_{\mathbf{n}}$ is

$$A_{\mathbf{n}} = \frac{\Gamma(a_1 + \sum\limits_{i=1}^{5} n_i)\Gamma(a_2 + \sum\limits_{i=1}^{5} n_i)\Gamma(a_3 + \sum\limits_{i=1}^{2} n_i)\Gamma(a_4 + \sum\limits_{i=1}^{2} n_i)\Gamma(a_5 + \sum\limits_{i=3}^{4} n_i)\Gamma(a_6 + \sum\limits_{i=3}^{4} n_i)}{\Big[\prod\limits_{i=1}^{5} n_i!\Gamma(b_i + n_i)\Big]\Gamma(b_6 + \sum\limits_{i=1}^{4} n_i)\Gamma(b_7 + \sum\limits_{i=1}^{4} n_i)} . \tag{48}$$

In order to proceed with our analysis, we define the auxiliary function

$$\Phi_{6B}(\mathbf{a},\ \mathbf{b} \,\Big|\, \mathbf{x},\ \mathbf{u},\ \mathbf{v}) = \mathbf{u}^{\mathbf{a}}\mathbf{v}^{\mathbf{b}-\mathbf{e}_7} T_{6B}(\mathbf{a},\ \mathbf{b} \,\Big|\, \mathbf{x}) \ , \tag{49}$$

with the intermediate variables $\mathbf{u} = (u_1, \cdots, u_6) = (1,\ 1,\ 1,\ 1,\ 1,\ 1)$, $\mathbf{v} = (v_1, \cdots, v_7)$, $\mathbf{v} = \mathbf{e}_7 = (1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1)$. Then one can obtain

$$\begin{aligned}
&\vartheta_{u_j} \Phi_{6B}(\mathbf{a},\ \mathbf{b} \,\Big|\, \mathbf{x},\ \mathbf{u},\ \mathbf{v}) = a_j \Phi_{6B}(\mathbf{a},\ \mathbf{b} \,\Big|\, \mathbf{x},\ \mathbf{u},\ \mathbf{v}) \ , \ (j = 1, \cdots, 6) \ , \\
&\vartheta_{v_k} \Phi_{6B}(\mathbf{a},\ \mathbf{b} \,\Big|\, \mathbf{x},\ \mathbf{u},\ \mathbf{v}) = (b_k - 1)\Phi_{6B}(\mathbf{a},\ \mathbf{b} \,\Big|\, \mathbf{x},\ \mathbf{u},\ \mathbf{v}) \ , \ (k = 1, \cdots, 7) \ .
\end{aligned} \tag{50}$$

Through the transformation

$$\begin{aligned}
&z_j = \frac{1}{u_j}, \quad (j = 1, \cdots, 6), \quad z_{6+k} = v_k, \quad (k = 1, \cdots, 7), \\
&z_{14} = \frac{x_1}{u_1 u_2 u_3 u_4 v_1 v_6 v_7}, \quad z_{15} = \frac{x_2}{u_1 u_2 u_3 u_4 v_2 v_6 v_7}, \\
&z_{16} = \frac{x_3}{u_1 u_2 u_5 u_6 v_3 v_6 v_7}, \quad z_{17} = \frac{x_4}{u_1 u_2 u_5 u_6 v_4 v_6 v_7}, \quad z_{18} = \frac{x_5}{u_1 u_2 v_5} \ ,
\end{aligned} \tag{51}$$

one derives the GKZ hypergeometric system for the four-loop vacuum diagram with six propagates for type B:

$$\mathbf{A_{6B}} \cdot \vec{\vartheta}_{6B} \Phi_{6B} = \mathbf{B_{6B}} \Phi_{6B} \ , \tag{52}$$

where

$$\begin{aligned}
&\mathbf{A_{6B}} = \begin{pmatrix} \mathbf{I}_{13\times 13} & \mathbf{A_{X6B}} \end{pmatrix} \ , \\
&\mathbf{A_{X6B}}^T = \begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & -1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0
\end{pmatrix} \ , \\
&\vec{\vartheta}_{6B}^T = (\vartheta_{z_1}, \cdots,\ \vartheta_{z_{18}}) \ , \\
&\mathbf{B_{6B}}^T = (-a_1,\ \cdots,\ -a_6,\ b_1 - 1,\ \cdots,\ b_7 - 1) \ .
\end{aligned} \tag{53}$$

Here, $\mathbf{I_{13\times13}}$ is a 13×13 unit matrix.

Correspondingly the dual matrix $\tilde{\mathbf{A}}_{\mathbf{6B}}$ of $\mathbf{A_{6B}}$ is

$$\tilde{\mathbf{A}}_{\mathbf{6B}} = \Big(-\mathbf{A_{X6B}}^{\;T} \quad \mathbf{I}_{5\times5} \Big) , \tag{54}$$

where $\mathbf{I}_{5\times5}$ is a 5×5 unit matrix. The row vectors of the matrix $\tilde{\mathbf{A}}_{\mathbf{6B}}$ induce the integer sublattice $\mathbf{B}$ which can be used to construct the formal solutions in hypergeometric series.

Defining the combined variables

$$\begin{aligned}
&y_1 = \frac{z_7 z_{12} z_{13} z_{14}}{z_1 z_2 z_3 z_4} \;, \quad y_2 = \frac{z_8 z_{12} z_{13} z_{15}}{z_1 z_2 z_3 z_4} \;, \\
&y_3 = \frac{z_9 z_{12} z_{13} z_{16}}{z_1 z_2 z_5 z_6} \;, \quad y_4 = \frac{z_{10} z_{12} z_{13} z_{17}}{z_1 z_2 z_5 z_6} \;, \quad y_5 = \frac{z_{11} z_{18}}{z_1 z_2} \;,
\end{aligned} \tag{55}$$

we write the solutions satisfying Eq. (52) as

$$\Phi_{6B}(\mathbf{z}) = \Big(\prod_{i=1}^{18} z_i^{\alpha_i} \Big) \varphi_{6B}(\mathbf{y}) \;, \tag{56}$$

where $\mathbf{y} = (y_1, \; \cdots, \; y_5)$, $\vec{\alpha}^{\,T} = (\alpha_1, \; \cdots, \; \alpha_{18})$ denotes a sequence of complex number such that

$$\mathbf{A_{6B}} \cdot \vec{\alpha} = \mathbf{B_{6B}} \;. \tag{57}$$

IV. GKZ HYPERGEOMETRIC SYSTEM OF THE FOUR-LOOP VACUUM INTEGRALS WITH SEVEN PROPAGATES

A. Four-loop vacuum diagram with seven propagates for type A

The general analytic expression for the Feynman integral of the four-loop vacuum diagram with seven propagates for type A in Fig. 4 can be written as

$$\begin{aligned}
U_{7A} = \Big(\Lambda^2_{\rm RE}\Big)^{8-2D} \int \frac{d^D \mathbf{q}}{(2\pi)^D} & \frac{1}{(q_1^2 - m_1^2)[(q_1 + q_3 + q_4)^2 - m_2^2](q_2^2 - m_3^2)} \\
& \times \frac{1}{[(q_2 + q_3 + q_4)^2 - m_4^2][(q_3 + q_4)^2 - m_5^2](q_3^2 - m_6^2)(q_4^2 - m_7^2)} .
\end{aligned} \tag{58}$$

Integrating out $\mathbf{q}$, the Mellin-Barnes representation of the Feynman integral of the four-loop vacuum diagram with seven propagates for type A can be written as

$$U_{7A} = \frac{-m_7^2}{(2\pi i)^6 (4\pi)^8} \Big(\frac{4\pi \Lambda^2_{\rm RE}}{m_7^2} \Big)^{8-2D} \int_{-i\infty}^{+i\infty} d\mathbf{s} \Big[\prod_{i=1}^{6} \Big(\frac{m_i^2}{m_7^2} \Big)^{s_i} \Gamma(-s_i) \Big]$$

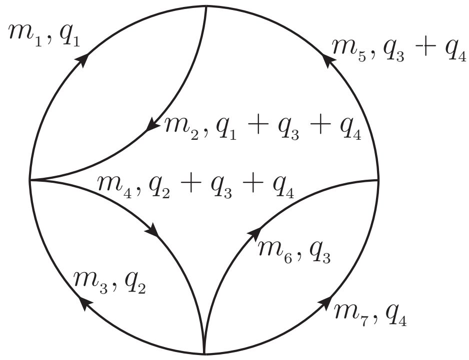

FIG. 4: Four-loop vacuum diagram with seven propagators for type A, which m_i denotes the mass of the i-th particle and q_j denotes the momentum.

$$\times\Big[\prod_{i=1}^{4}\Gamma(\frac{D}{2}-1-s_i)\Big]\Gamma(\frac{D}{2}-1-s_6)\Gamma(6-\frac{3D}{2}+\sum_{i=1}^{6}s_i)\Gamma(7-2D+\sum_{i=1}^{6}s_i)$$
$$\times\frac{\Gamma(\frac{3D}{2}-5-\sum\limits_{i=1}^{5}s_i)\Gamma(2-\frac{D}{2}+s_1+s_2)\Gamma(2-\frac{D}{2}+s_3+s_4)\Gamma(1+s_5)}{\Gamma(5-D+\sum\limits_{i=1}^{5}s_i)\Gamma(D-2-s_1-s_2)\Gamma(D-2-s_3-s_4)}\,, \tag{59}$$

where $\mathbf{s}=(s_1,\ \cdots,\ s_6)$.

Taking the residue of the pole of $\Gamma(-s_i)$, $(i=1,\cdots,6)$, one can derive one linear independent term of the integral:

$$U_{7A}\ni\frac{-m_7^2}{(4\pi)^8}\Big(\frac{4\pi\Lambda_{\rm RE}^2}{m_7^2}\Big)^{8-2D}\frac{\pi^4\sin^2\pi D}{\sin^5\frac{\pi D}{2}\sin\frac{3\pi D}{2}}T_{7A}(\mathbf{a},\ \mathbf{b}\ \Big|\ \mathbf{x})\ , \tag{60}$$

with

$$T_{7A}(\mathbf{a},\ \mathbf{b}\ \Big|\ \mathbf{x})=\sum_{\mathbf{n}=\mathbf{0}}^{\infty}A_{\mathbf{n}}\mathbf{x}^{\mathbf{n}}, \tag{61}$$

where $\mathbf{n}=(n_1,\ \cdots,\ n_6)$, $\mathbf{x}=(x_1,\ \cdots,\ x_6)$, $x_i=\frac{m_i^2}{m_7^2}$, $\mathbf{a}=(a_1,\ \cdots,a_7)$ and $\mathbf{b}=(b_1,\ \cdots,b_7)$ with

$$a_1=6-\frac{3D}{2},\ a_2=7-2D,\ a_3=a_5=2-\frac{D}{2},\ a_4=a_6=3-D,\ a_7=1\ ,$$
$$b_1=b_2=b_3=b_4=b_5=2-\frac{D}{2},\ b_6=5-D,\ b_7=6-\frac{3D}{2}\ , \tag{62}$$

and the coefficient $A_{\mathbf{n}}$ is

$$A_{\mathbf{n}}=\Gamma(a_1+\sum_{i=1}^{6}n_i)\Gamma(a_2+\sum_{i=1}^{6}n_i)\Gamma(a_3+\sum_{i=1}^{2}n_i)$$

$$\times \frac{\Gamma(a_4+\sum_{i=1}^{2}n_i)\Gamma(a_5+\sum_{i=3}^{4}n_i)\Gamma(a_6+\sum_{i=3}^{4}n_i)\Gamma(a_7+n_5)}{\Big[\prod_{i=1}^{6}n_i!\Big]\Big[\prod_{i=1}^{4}\Gamma(b_i+n_i)\Big]\Gamma(b_5+n_6)\Gamma(b_6+\sum_{i=1}^{5}n_i)\Gamma(b_7+\sum_{i=1}^{5}n_i)} . \tag{63}$$

In order to proceed with our analysis, we define the auxiliary function

$$\Phi_{7A}(\mathbf{a},\ \mathbf{b}\,\Big|\,\mathbf{x},\ \mathbf{u},\ \mathbf{v}) = \mathbf{u}^{\mathbf{a}}\mathbf{v}^{\mathbf{b}-\mathbf{e}_7}T_{7A}(\mathbf{a},\ \mathbf{b}\,\Big|\,\mathbf{x})\ , \tag{64}$$

with the intermediate variables $\mathbf{u} = (u_1,\cdots,u_7)$, $\mathbf{v} = (v_1,\cdots,v_7)$, $\mathbf{u} = \mathbf{v} = \mathbf{e}_7 = (1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1)$. Then one can obtain

$$\begin{aligned}
&\vartheta_{u_j}\Phi_{7A}(\mathbf{a},\ \mathbf{b}\,\Big|\,\mathbf{x},\ \mathbf{u},\ \mathbf{v}) = a_j\Phi_{7A}(\mathbf{a},\ \mathbf{b}\,\Big|\,\mathbf{x},\ \mathbf{u},\ \mathbf{v})\ ,\ (j=1,\cdots,7)\ ,\\
&\vartheta_{v_k}\Phi_{7A}(\mathbf{a},\ \mathbf{b}\,\Big|\,\mathbf{x},\ \mathbf{u},\ \mathbf{v}) = (b_k-1)\Phi_{7A}(\mathbf{a},\ \mathbf{b}\,\Big|\,\mathbf{x},\ \mathbf{u},\ \mathbf{v})\ ,\ (k=1,\cdots,7)\ .
\end{aligned} \tag{65}$$

Through the transformation

$$\begin{aligned}
&z_j = \frac{1}{u_j},\ \ (j=1,\cdots,7),\ \ z_{7+k} = v_k,\ \ (k=1,\cdots,7),\\
&z_{15} = \frac{x_1}{u_1u_2u_3u_4v_1v_6v_7},\ \ z_{16} = \frac{x_2}{u_1u_2u_3u_4v_2v_6v_7},\\
&z_{17} = \frac{x_3}{u_1u_2u_5u_6v_3v_6v_7},\ \ z_{18} = \frac{x_4}{u_1u_2u_5u_6v_4v_6v_7},\\
&z_{19} = \frac{x_5}{u_1u_2u_7v_6v_7}\ ,\ \ z_{20} = \frac{x_6}{u_1u_2v_5}\ ,
\end{aligned} \tag{66}$$

one derives the GKZ hypergeometric system for the four-loop vacuum diagram with seven propagates for type A:

$$\mathbf{A_{7A}}\cdot\vec{\vartheta}_{7A}\Phi_{7A} = \mathbf{B_{7A}}\Phi_{7A}\ , \tag{67}$$

where

$$\begin{aligned}
&\mathbf{A_{7A}} = \begin{pmatrix} \mathbf{I_{14\times14}} & \mathbf{A_{X7A}} \end{pmatrix}\ ,\\
&\mathbf{A_{X7A}}^{\,T} = \begin{pmatrix}
1&1&1&1&0&0&0&-1&0&0&0&0&-1&-1\\
1&1&1&1&0&0&0&0&-1&0&0&0&-1&-1\\
1&1&0&0&1&1&0&0&0&-1&0&0&-1&-1\\
1&1&0&0&1&1&0&0&0&0&-1&0&-1&-1\\
1&1&0&0&0&0&1&0&0&0&0&0&-1&-1\\
1&1&0&0&0&0&0&0&0&0&0&-1&0&0
\end{pmatrix},\\
&\vec{\vartheta}_{7A}^{\,T} = (\vartheta_{z_1},\cdots,\ \vartheta_{z_{20}})\ ,\\
&\mathbf{B_{7A}}^{\,T} = (-a_1,\ \cdots,\ -a_7,\ b_1-1,\ \cdots,\ b_7-1)\ .
\end{aligned} \tag{68}$$

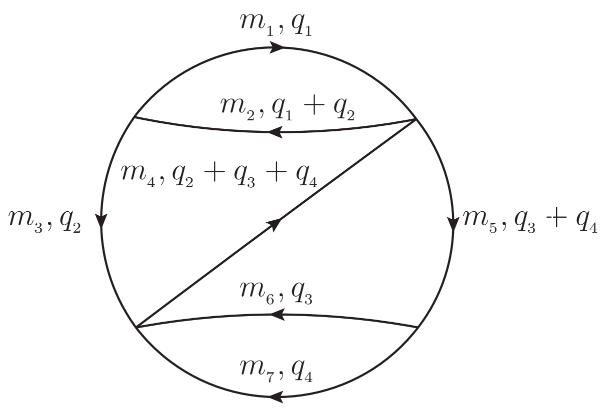

FIG. 5: Four-loop vacuum diagram with seven propagators for type B, which m_i denotes the mass of the i-th particle and q_j denotes the momentum.

Here, $\mathbf{I_{14\times 14}}$ is a 14×14 unit matrix.

Correspondingly the dual matrix $\tilde{\mathbf{A}}_{\mathbf{7A}}$ of $\mathbf{A_{7A}}$ is

$$\tilde{\mathbf{A}}_{\mathbf{7A}} = \Big(-\mathbf{A_{X7A}}^{\;T} \quad \mathbf{I_{6\times 6}} \Big) , \tag{69}$$

where $\mathbf{I_{6\times 6}}$ is a 6×6 unit matrix. The row vectors of the matrix $\tilde{\mathbf{A}}_{\mathbf{7A}}$ induce the integer sublattice $\mathbf{B}$ which can be used to construct the formal solutions in hypergeometric series.

Defining the combined variables

$$\begin{aligned} y_1 &= \frac{z_8 z_{13} z_{14} z_{15}}{z_1 z_2 z_3 z_4} \;, \quad y_2 = \frac{z_9 z_{13} z_{14} z_{16}}{z_1 z_2 z_3 z_4} \;, \quad y_3 = \frac{z_{10} z_{13} z_{14} z_{17}}{z_1 z_2 z_5 z_6} \;, \\ y_4 &= \frac{z_{11} z_{13} z_{14} z_{18}}{z_1 z_2 z_5 z_6} \;, \quad y_5 = \frac{z_{13} z_{14} z_{19}}{z_1 z_2 z_7} \;, \quad y_6 = \frac{z_{12} z_{20}}{z_1 z_2} \;, \end{aligned} \tag{70}$$

we write the solutions satisfying Eq. (67) as

$$\Phi_{7A}(\mathbf{z}) = \Big(\prod_{i=1}^{20} z_i^{\alpha_i} \Big) \varphi_{7A}(\mathbf{y}) \;, \tag{71}$$

where $\mathbf{y} = (y_1,\ \cdots,\ y_6)$, and $\vec{\alpha}^{\,T} = (\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_{20})$ denotes a sequence of complex number such that

$$\mathbf{A_{7A}} \cdot \vec{\alpha} = \mathbf{B_{7A}} \;. \tag{72}$$

B. Four-loop vacuum diagram with seven propagates for type B

The general analytic expression for the Feynman integral of the four-loop vacuum diagram with seven propagates for type B in Fig. 5 is

$$U_{_{7B}} = \Big(\Lambda^2_{_{\rm RE}}\Big)^{8-2D}\int\frac{d^D\mathbf{q}}{(2\pi)^D}\frac{1}{(q_{_1}^2-m_{_1}^2)[(q_{_1}+q_{_2})^2-m_{_2}^2](q_{_2}^2-m_{_3}^2)}$$
$$\times\frac{1}{[(q_{_2}+q_{_3}+q_{_4})^2-m_{_4}^2][(q_{_3}+q_{_4})^2-m_{_5}^2](q_{_3}^2-m_{_6}^2)(q_{_4}^2-m_{_7}^2)}. \tag{73}$$

Integrating out $\mathbf{q}$, the Mellin-Barnes representation of the Feynman integral of the four-loop vacuum diagram with seven propagates for type B can be written as

$$U_{_{7B}} = \frac{-m_{_7}^2}{(2\pi i)^6(4\pi)^8}\Big(\frac{4\pi\Lambda^2_{_{\rm RE}}}{m_{_7}^2}\Big)^{8-2D}\int_{-i\infty}^{+i\infty}d\mathbf{s}\Big[\prod_{i=1}^6\Big(\frac{m_{_i}^2}{m_{_7}^2}\Big)^{s_{_i}}\Gamma(-s_{_i})\Big]\Big[\prod_{i=1}^2\Gamma(\frac{D}{2}-1-s_{_i})\Big]$$
$$\times\Gamma(\frac{D}{2}-1-s_{_4})\Gamma(\frac{D}{2}-1-s_{_6})\Gamma(6-\frac{3D}{2}+\sum_{i=1}^6 s_{_i})\Gamma(7-2D+\sum_{i=1}^6 s_{_i})\Gamma(4-D+\sum_{i=1}^4 s_{_i})$$
$$\times\frac{\Gamma(\frac{3D}{2}-5-\sum\limits_{i=1}^5 s_{_i})\Gamma(D-3-\sum\limits_{i=1}^3 s_{_i})\Gamma(2-\frac{D}{2}+s_{_1}+s_{_2})\Gamma(1+s_{_3})\Gamma(1+s_{_5})}{\Gamma(D-2-s_{_1}-s_{_2})\Gamma(\frac{3D}{2}-4-\sum\limits_{i=1}^4 s_{_i})\Gamma(3-\frac{D}{2}+\sum\limits_{i=1}^3 s_{_i})\Gamma(5-D+\sum\limits_{i=1}^5 s_{_i})}\;, \tag{74}$$

where $\mathbf{s}=(s_{_1},\;\cdots,\;s_{_6})$.

Taking the residue of the pole of $\Gamma(-s_{_i})$, $(i=1,\cdots,6)$, one can derive one linear independent term of the integral:

$$U_{_{7B}} \ni \frac{m_{_7}^2}{(4\pi)^8}\Big(\frac{4\pi\Lambda^2_{_{\rm RE}}}{m_{_7}^2}\Big)^{8-2D}\frac{\pi^4}{\sin^4\frac{\pi D}{2}}T_{_{7B}}(\mathbf{a},\;\mathbf{b}\;\big|\;\mathbf{x})\;, \tag{75}$$

with

$$T_{_{7B}}(\mathbf{a},\;\mathbf{b}\;\big|\;\mathbf{x}) = \sum_{\mathbf{n}=\mathbf{0}}^{\infty}A_{_\mathbf{n}}\mathbf{x}^{\mathbf{n}}, \tag{76}$$

where $\mathbf{n}=(n_{_1},\;\cdots,\;n_{_6})$, $\mathbf{x}=(x_{_1},\;\cdots,\;x_{_6})$, $x_{_i}=\frac{m_{_i}^2}{m_{_7}^2}$, $\mathbf{a}=(a_{_1},\;\cdots,a_{_8})$ and $\mathbf{b}=(b_{_1},\;\cdots,b_{_8})$ with

$$a_{_1}=6-\frac{3D}{2},\;a_{_2}=7-2D,\;a_{_3}=5-\frac{3D}{2},\;a_{_4}=4-D,$$
$$a_{_5}=2-\frac{D}{2},\;a_{_6}=3-D,\;a_{_7}=a_{_8}=1,\;b_{_1}=b_{_2}=b_{_3}=b_{_4}=2-\frac{D}{2},$$
$$b_{_5}=3-\frac{D}{2},\;b_{_6}=4-D,\;b_{_7}=5-D,\;b_{_8}=6-\frac{3D}{2}\;, \tag{77}$$

and the coefficient $A_{\mathbf{n}}$ is

$$A_{\mathbf{n}} = \frac{\Gamma(a_1 + \sum\limits_{i=1}^{6} n_i)\Gamma(a_2 + \sum\limits_{i=1}^{6} n_i)\Gamma(a_3 + \sum\limits_{i=1}^{4} n_i)\Gamma(a_4 + \sum\limits_{i=1}^{4} n_i)}{\Big[\prod\limits_{i=1}^{6} n_i!\Big]\Gamma(b_1 + n_1)\Gamma(b_2 + n_2)\Gamma(b_3 + n_4)\Gamma(b_4 + n_6)}$$
$$\times \frac{\Gamma(a_5 + \sum\limits_{i=1}^{2} n_i)\Gamma(a_6 + \sum\limits_{i=1}^{2} n_i)\Gamma(a_7 + n_3)\Gamma(a_8 + n_5)}{\Gamma(b_5 + \sum\limits_{i=1}^{3} n_i)\Gamma(b_6 + \sum\limits_{i=1}^{3} n_i)\Gamma(b_7 + \sum\limits_{i=1}^{5} n_i)\Gamma(b_8 + \sum\limits_{i=1}^{5} n_i)} . \tag{78}$$

We also define the auxiliary function

$$\Phi_{7B}(\mathbf{a},\ \mathbf{b}\ \Big|\ \mathbf{x},\ \mathbf{u},\ \mathbf{v}) = \mathbf{u}^{\mathbf{a}}\mathbf{v}^{\mathbf{b}-\mathbf{e}_8} T_{7B}(\mathbf{a},\ \mathbf{b}\ \Big|\ \mathbf{x})\ , \tag{79}$$

with the intermediate variables $\mathbf{u} = (u_1, \cdots, u_8)$, $\mathbf{v} = (v_1, \cdots, v_8)$, $\mathbf{u} = \mathbf{v} = \mathbf{e}_8 = (1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1)$. Then one can obtain

$$\begin{aligned}
&\vartheta_{u_j}\Phi_{7B}(\mathbf{a},\ \mathbf{b}\ \Big|\ \mathbf{x},\ \mathbf{u},\ \mathbf{v}) = a_j\Phi_{7B}(\mathbf{a},\ \mathbf{b}\ \Big|\ \mathbf{x},\ \mathbf{u},\ \mathbf{v})\ ,\ (j = 1, \cdots, 8)\ , \\
&\vartheta_{v_k}\Phi_{7B}(\mathbf{a},\ \mathbf{b}\ \Big|\ \mathbf{x},\ \mathbf{u},\ \mathbf{v}) = (b_k - 1)\Phi_{7B}(\mathbf{a},\ \mathbf{b}\ \Big|\ \mathbf{x},\ \mathbf{u},\ \mathbf{v})\ ,\ (k = 1, \cdots, 8)\ .
\end{aligned} \tag{80}$$

Through the transformation

$$\begin{aligned}
&z_j = \frac{1}{u_j},\quad (j = 1, \cdots, 8),\quad z_{8+k} = v_k,\quad (k = 1, \cdots, 8), \\
&z_{17} = \frac{x_1}{u_1u_2u_3u_4u_5u_6v_1v_5v_6v_7v_8},\quad z_{18} = \frac{x_2}{u_1u_2u_3u_4u_5u_6v_2v_5v_6v_7v_8}, \\
&z_{19} = \frac{x_3}{u_1u_2u_3u_4u_7v_5v_6v_7v_8},\quad z_{20} = \frac{x_4}{u_1u_2u_3u_4v_3v_7v_8}, \\
&z_{21} = \frac{x_5}{u_1u_2u_8v_7v_8}\ ,\quad z_{22} = \frac{x_6}{u_1u_2v_4}\ ,
\end{aligned} \tag{81}$$

one derives the GKZ hypergeometric system for the four-loop vacuum diagram with seven propagates for type B:

$$\mathbf{A_{7B}} \cdot \vec{\vartheta}_{7B}\Phi_{7B} = \mathbf{B_{7B}}\Phi_{7B}\ , \tag{82}$$

where

$$\mathbf{A_{7B}} = \begin{pmatrix} \mathbf{I_{16\times16}} & \mathbf{A_{X7B}} \end{pmatrix}\ ,$$

$$
\begin{aligned}
&\mathbf{A_{X7B}}^T = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} , \\
&\vec{\vartheta}_{7B}^T = (\vartheta_{z_1}, \cdots, \vartheta_{z_{22}}) , \\
&\mathbf{B_{7B}}^T = (-a_1, \; \cdots, \; -a_8, \; b_1 - 1, \; \cdots, \; b_8 - 1) . \qquad (83)
\end{aligned}
$$

Here, $\mathbf{I_{16\times16}}$ is a 16×16 unit matrix.

Correspondingly the dual matrix $\tilde{\mathbf{A}}_{\mathbf{7B}}$ of $\mathbf{A_{7B}}$ is

$$
\tilde{\mathbf{A}}_{\mathbf{7B}} = \Big(-\mathbf{A_{X7B}}^T \quad \mathbf{I_{6\times6}} \Big) , \qquad (84)
$$

where $\mathbf{I_{6\times6}}$ is a 6×6 unit matrix. The row vectors of the matrix $\tilde{\mathbf{A}}_{\mathbf{7B}}$ also induce the integer sublattice $\mathbf{B}$ which can be used to construct the formal solutions in hypergeometric series.

Defining the combined variables

$$
\begin{aligned}
&y_1 = \frac{z_9 z_{13} z_{14} z_{15} z_{16} z_{17}}{z_1 z_2 z_3 z_4 z_5 z_6} , \quad y_2 = \frac{z_{10} z_{13} z_{14} z_{15} z_{16} z_{18}}{z_1 z_2 z_3 z_4 z_5 z_6} , \\
&y_3 = \frac{z_{13} z_{14} z_{15} z_{16} z_{19}}{z_1 z_2 z_3 z_4 z_7} , \quad y_4 = \frac{z_{11} z_{15} z_{16} z_{20}}{z_1 z_2 z_3 z_4} , \\
&y_5 = \frac{z_{15} z_{16} z_{21}}{z_1 z_2 z_8} , \quad y_6 = \frac{z_{12} z_{22}}{z_1 z_2} , \qquad (85)
\end{aligned}
$$

we write the solutions satisfying Eq. (82) as

$$
\Phi_{7B}(\mathbf{z}) = \Big(\prod_{i=1}^{22} z_i^{\alpha_i} \Big) \varphi_{7B}(\mathbf{y}) , \qquad (86)
$$

where $\vec{\alpha}^T = (\alpha_1, \; \cdots, \; \alpha_{22})$ denotes a sequence of complex number such that

$$
\mathbf{A_{7B}} \cdot \vec{\alpha} = \mathbf{B_{7B}} . \qquad (87)
$$

V. THE HYPERGEOMETRIC SERIES SOLUTIONS OF THE FOUR-LOOP VACUUM INTEGRALS

A. The general case of the four-loop vacuum integral with five propagates

In this subsection, we will show the hypergeometric series solutions of the GKZ hypergeometric system of the four-loop vacuum integral with five propagates with arbitrary masses. To construct the hypergeometric series solutions of the GKZ hypergeometric system of the four-loop vacuum integral with five propagates in Eq. (18) is equivalent to choose a set of the linear independent column vectors of the matrix in Eq. (20) which spans the dual space. We denote the submatrix composed of the first, third, fourth and fifth column vectors of the dual matrix of Eq. (20) as $\tilde{\mathbf{A}}_{1345}$, i.e.

$$\tilde{\mathbf{A}}_{1345} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix} . \tag{88}$$

Obviously $\det \tilde{\mathbf{A}}_{1345} = 1 \neq 0$, and

$$\begin{aligned} \mathbf{B}_{1345} &= \tilde{\mathbf{A}}_{1345}^{-1} \cdot \tilde{\mathbf{A}}_{5} \\ &= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 \end{pmatrix} . \end{aligned} \tag{89}$$

Taking 4 row vectors of the matrix $\mathbf{B}_{1345}$ as the basis of integer lattice, one constructs the GKZ hypergeometric series solutions in parameter space through choosing the sets of column indices $I_i \subset [1, \cdots, 10]$ $(i = 1, \cdots, 16)$ which are consistent with the basis of integer lattice $\mathbf{B}_{1345}$.

We take the set of column indices $I_1 = [2, 6, \cdots, 10]$, i.e. the implement $J_1 = [1, \cdots, 10] \setminus I_1 = [1, 3, 4, 5]$. The choice on the set of indices implies the exponent numbers $\alpha_1 = \alpha_3 = \alpha_4 = \alpha_5 = 0$. Through Eq. (24), one can have

$$\alpha_2 = a_1 - a_2 ,\ \alpha_6 = \sum_{i=1}^{4} b_i - a_1 - 4 ,\ \alpha_7 = 1 - b_1 ,$$

$$\alpha_8 = 1 - b_2,\ \alpha_9 = 1 - b_3,\ \alpha_{10} = \sum_{i=1}^{3} b_i - a_1 - 3\ . \tag{90}$$

Combined with Eq. (9), we can have

$$\alpha_2 = \frac{D}{2} - 1,\ \alpha_6 = -\frac{D}{2},\ \alpha_7 = \alpha_8 = \alpha_9 = \frac{D}{2} - 1,\ \alpha_{10} = -1\ . \tag{91}$$

According the basis of integer lattice $\mathbf{B}_{1345}$, the corresponding hypergeometric series solution with quadruple independent variables is written as

$$\begin{aligned}\Phi^{(1)}_{[1345]} &= \prod_{i=1}^{10} z_i^{\alpha_i} \sum_{\mathbf{n}=0}^{\infty} c^{(1)}_{[1345]}(\mathbf{n}) \Big(\frac{z_1 z_2}{z_6 z_{10}}\Big)^{n_1} \Big(\frac{z_3 z_7}{z_6 z_{10}}\Big)^{n_2} \Big(\frac{z_4 z_8}{z_6 z_{10}}\Big)^{n_3} \Big(\frac{z_5 z_9}{z_6 z_{10}}\Big)^{n_4} \\ &= \prod_{i=1}^{10} z_i^{\alpha_i} \sum_{\mathbf{n}=0}^{\infty} c^{(1)}_{[1345]}(\mathbf{n}) \Big(\frac{1}{y_4}\Big)^{n_1} \Big(\frac{y_1}{y_4}\Big)^{n_2} \Big(\frac{y_2}{y_4}\Big)^{n_3} \Big(\frac{y_3}{y_4}\Big)^{n_4}\ ,\end{aligned} \tag{92}$$

where the coefficient is

$$\begin{aligned}c^{(1)}_{[1345]}(\mathbf{n}) = \Big\{ \Big[\prod_{i=1}^{4} n_i ! \Big] &\Gamma(1 + \alpha_2 + n_1) \Gamma(1 + \alpha_6 - \sum_{i=1}^{4} n_i) \Gamma(1 + \alpha_7 + n_2) \\ &\times \Gamma(1 + \alpha_8 + n_3) \Gamma(1 + \alpha_9 + n_4) \Gamma(1 + \alpha_{10} - \sum_{i=1}^{4} n_i) \Big\}^{-1}\ .\end{aligned} \tag{93}$$

Using the relation in Eq. (6), one can have

$$c^{(1)}_{[1345]}(\mathbf{n}) \propto \frac{\Gamma(-\alpha_6 + \sum\limits_{i=1}^{4} n_i) \Gamma(-\alpha_{10} + \sum\limits_{i=1}^{4} n_i)}{\Big[\prod\limits_{i=1}^{4} n_i ! \Big] \Gamma(1 + \alpha_2 + n_1) \Gamma(1 + \alpha_7 + n_2) \Gamma(1 + \alpha_8 + n_3) \Gamma(1 + \alpha_9 + n_4)}, \tag{94}$$

where we ignore the constant coefficient term $\frac{\sin \pi\alpha_6 \sin \pi\alpha_{10}}{\pi^2}$. And then, through Eq. (91), the corresponding hypergeometric series solution can be written as

$$\begin{aligned}\Phi^{(1)}_{[1345]} &= y_1^{\frac{D}{2}-1} y_2^{\frac{D}{2}-1} y_3^{\frac{D}{2}-1} y_4^{-1} \sum_{\mathbf{n}=0}^{\infty} c^{(1)}_{[1345]}(\mathbf{n}) f_{[1345]}\ , \\ f_{[1345]} &= \Big(\frac{1}{y_4}\Big)^{n_1} \Big(\frac{y_1}{y_4}\Big)^{n_2} \Big(\frac{y_2}{y_4}\Big)^{n_3} \Big(\frac{y_3}{y_4}\Big)^{n_4}\ ,\end{aligned} \tag{95}$$

with the coefficient is

$$c^{(1)}_{[1345]}(\mathbf{n}) = \frac{\Gamma(\frac{D}{2} + \sum\limits_{i=1}^{4} n_i) \Gamma(1 + \sum\limits_{i=1}^{4} n_i)}{\prod\limits_{i=1}^{4} n_i ! \Gamma(\frac{D}{2} + n_i)}\ . \tag{96}$$

Here, the convergent region of the hypergeometric function $\Phi^{(1)}_{[1345]}$ in Eq. (95) is

$$\Xi_{[1345]} = \{ (y_1,\ y_2,\ y_3,\ y_4) \Big| 1 < |y_4|,\ |y_1| < |y_4|,\ |y_2| < |y_4|,\ |y_3| < |y_4| \}\ , \tag{97}$$

which shows that $\Phi^{(1)}_{[1345]}$ is in neighborhood of regular singularity ∞.

According the basis of integer lattice $\mathbf{B}_{1345}$, we can also obtain other fifteen hypergeometric solutions, which the expressions are collected in Appendix B. The sixteen hypergeometric series solutions $\Phi^{(i)}_{[1345]}$ whose convergent region is $\Xi_{[1345]}$ can constitute a fundamental solution system. The combination coefficients are determined by the value of the scalar integral of an ordinary point or some regular singularities.

Multiplying one of the row vectors of the matrix $\mathbf{B}_{1345}$ by -1, the induced integer matrix can also be chosen as a basis of the integer lattice space of certain hypergeometric series. Taking 4 row vectors of the following matrix as the basis of integer lattice,

$$\begin{aligned}\mathbf{B}_{\tilde{1}345} &= \mathrm{diag}(-1,1,1,1)\cdot\mathbf{B}_{1345}\\ &= \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 \end{pmatrix},\end{aligned} \tag{98}$$

one obtains sixteen hypergeometric series solutions $\Phi^{(i)}_{[\tilde{1}345]}$ $(i=1,\cdots,16)$ similarly, which the expressions are collected in Appendix C. The convergent region of the hypergeometric functions $\Phi^{(i)}_{[\tilde{1}345]}$ is

$$\Xi_{[\tilde{1}345]} = \{(y_1,\ y_2,\ y_3,\ y_4)\big|\,|y_1|<1,\ |y_2|<1,\ |y_3|<1,\ |y_4|<1\}\ , \tag{99}$$

which shows that $\Phi^{(i)}_{[\tilde{1}345]}$ are in neighborhood of regular singularity 0 and can constitute a fundamental solution system.

Taking 4 row vectors of the following matrix as the basis of integer lattice,

$$\begin{aligned}\mathbf{B}_{1\tilde{3}45} &= \mathrm{diag}(1,-1,1,1)\cdot\mathbf{B}_{1345}\\ &= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 \end{pmatrix},\end{aligned} \tag{100}$$

one also obtains sixteen hypergeometric series solutions $\Phi^{(i)}_{[1\tilde{3}45]}$ $(i=1,\cdots,16)$:

$$\Phi^{(i)}_{[1\tilde{3}45]} = \Phi^{(i)}_{[1345]}(y_4 \leftrightarrow y_1)\ , \tag{101}$$

which the expressions can be obtained by interchanging between y_4 and y_1 in $\Phi^{(i)}_{[1345]}$. The convergent region of the hypergeometric functions $\Phi^{(i)}_{[1\tilde{3}45]}$ is

$$\Xi_{[1\tilde{3}45]} = \{(y_1,\ y_2,\ y_3,\ y_4)\Big|1 < |y_1|,\ |y_4| < |y_1|,\ |y_2| < |y_1|,\ |y_3| < |y_1|\}\ , \tag{102}$$

which shows that $\Phi^{(i)}_{[1\tilde{3}45]}$ are in neighborhood of regular singularity ∞ and can constitute a fundamental solution system.

Taking 4 row vectors of the following matrix as the basis of integer lattice,

$$\begin{aligned}\mathbf{B}_{13\tilde{4}5} &= \mathrm{diag}(1,1,-1,1)\cdot\mathbf{B}_{1345} \\ &= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 \end{pmatrix}\ ,\end{aligned} \tag{103}$$

one obtains sixteen hypergeometric series solutions $\Phi^{(i)}_{[13\tilde{4}5]}$ $(i = 1, \cdots, 16)$:

$$\Phi^{(i)}_{[13\tilde{4}5]} = \Phi^{(i)}_{[1345]}(y_4 \leftrightarrow y_2)\ , \tag{104}$$

which the expressions can be obtained by interchanging between y_4 and y_2 in $\Phi^{(i)}_{[1345]}$. The convergent region of the hypergeometric functions $\Phi^{(i)}_{[13\tilde{4}5]}$ is

$$\Xi_{[13\tilde{4}5]} = \{(y_1,\ y_2,\ y_3,\ y_4)\Big|1 < |y_2|,\ |y_1| < |y_2|,\ |y_4| < |y_2|,\ |y_3| < |y_2|\}\ , \tag{105}$$

which shows that $\Phi^{(i)}_{[13\tilde{4}5]}$ are in neighborhood of regular singularity ∞ and can constitute a fundamental solution system.

Taking 4 row vectors of the following matrix as the basis of integer lattice,

$$\begin{aligned}\mathbf{B}_{134\tilde{5}} &= \mathrm{diag}(1,1,1,-1)\cdot\mathbf{B}_{1345} \\ &= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 \end{pmatrix}\ ,\end{aligned} \tag{106}$$

one obtains sixteen hypergeometric series solutions $\Phi^{(i)}_{[134\tilde{5}]}$ $(i = 1, \cdots, 16)$:

$$\Phi^{(i)}_{[134\tilde{5}]} = \Phi^{(i)}_{[1345]}(y_4 \leftrightarrow y_3)\ , \tag{107}$$

which the expressions can be obtained by interchanging between y_4 and y_3 in $\Phi^{(i)}_{[1345]}$. The convergent region of the hypergeometric functions $\Phi^{(i)}_{[134\tilde{5}]}$ is

$$\Xi_{[134\tilde{5}]} = \{(y_1,\ y_2,\ y_3,\ y_4)\Big|1 < |y_3|,\ |y_1| < |y_3|,\ |y_2| < |y_3|,\ |y_4| < |y_3|\}\ , \tag{108}$$

which shows that $\Phi^{(i)}_{[134\tilde{5}]}$ are in neighborhood of regular singularity ∞ and can constitute a fundamental solution system.

B. The special case of the four-loop vacuum integral with five propagates

In order to elucidate how to obtain the analytical expression clearly, we assume the two nonzero virtual mass for the four-loop vacuum integral with five propagates. The corresponding integral of the special case for the four-loop vacuum diagram can be expressed as a linear combination of those corresponding functionally independent Gauss functions.

Through Sec. II, the GKZ hypergeometric system in this special case ($m_1 \neq 0,\ m_5 \neq 0,\ m_2 = m_3 = m_4 = 0$) can be simplified as

$$\mathbf{A_{51}} \cdot \vec{\vartheta}_{51}\Phi_{51} = \mathbf{B_5}\Phi_{51}\ , \tag{109}$$

where the vector of Euler operators is defined as

$$\vec{\vartheta}^{\,T}_{51} = (\vartheta_{z_1}, \cdots,\ \vartheta_{z_7})\ , \tag{110}$$

and the matrix $\mathbf{A}_{51}$ is obtained through deleting the 8th, 9th, and 10th columns of the matrix $\mathbf{A_5}$:

$$\mathbf{A}_{51} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}\ . \tag{111}$$

Through Eq. (109), one can have the relations

$$\alpha_1 + \alpha_7 = -a_1\ , \quad \alpha_2 + \alpha_7 = -a_2\ , \quad \alpha_3 - \alpha_7 = b_1 - 1\ , \tag{112}$$

with $a_1 = 4 - \frac{3D}{2}$, $a_2 = 5 - 2D$, $b_1 = 2 - \frac{D}{2}$, and the other α_i are zero. The dual matrix $\tilde{\mathbf{A}}_{51}$ of $\mathbf{A}_{51}$ is

$$\tilde{\mathbf{A}}_{51} = \begin{pmatrix} -1 & -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{113}$$

The integer sublattice $\mathbf{B_{51}}$ is determined by the dual matrix $\tilde{\mathbf{A}}_{51}$ with $\mathbf{B_{51}} = \tilde{\mathbf{A}}_{51}$. The integer sublattice $\mathbf{B_{51}}$ implies that the system of fundamental solutions is composed by two linear independent hypergeometric functions. We take the set of column indices $I_1 = [1, \cdots, 6]$, which implies the exponent numbers $\alpha_7 = 0$. And then, the corresponding hypergeometric series solution can be written as

$$\Phi^{(1)}_{[51]}(y_1) = {}_2F_1 \left(\begin{array}{cc} 4 - \frac{3D}{2}, & 5 - 2D \\ & 2 - \frac{D}{2} \end{array} \middle| y_1 \right) , \tag{114}$$

with $y_1 = x_1 = m_1^2/m_5^2$, and ${}_2F_1$ is Gauss function:

$${}_2F_1 \left(\begin{array}{cc} a, & b \\ & c \end{array} \middle| x \right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n! (c)_n} x^n , \tag{115}$$

with $(a)_n = \Gamma(a+n)/\Gamma(a)$. We also can take the set of column indices $I_2 = [1, 2, 4, \cdots, 7]$, which implies the exponent numbers $\alpha_3 = 0$. And then, the another hypergeometric series solution can be written as

$$\Phi^{(2)}_{[51]}(y_1) = (y_1)^{D/2-1} {}_2F_1 \left(\begin{array}{cc} 3 - D, & 4 - \frac{3D}{2} \\ & \frac{D}{2} \end{array} \middle| y_1 \right) . \tag{116}$$

Here, the convergent region of the hypergeometric functions $\Phi^{(1,2)}_{[51]}(y_1)$ is $|y_1| < 1$. In the region $|y_1| < 1$, the integral correspondingly is a linear combination of two fundamental solutions:

$$\Phi_{51}(y_1) = C^{(1)}_{[51]} \Phi^{(1)}_{[51]}(y_1) + C^{(2)}_{[51]} \Phi^{(2)}_{[51]}(y_1) . \tag{117}$$

Multiplying one of the row vectors of the integer matrix $\mathbf{B}_{51}$ by -1, the induced integer matrix also can be chosen as a basis of the integer lattice space of certain hypergeometric series. And the corresponding system of fundamental solutions for is similarly composed by two Gauss functions:

$$\begin{aligned} \Phi^{(3)}_{[51]}(y_1) &= (y_1)^{\frac{3D}{2}-4} {}_2F_1 \left(\begin{array}{cc} 4 - \frac{3D}{2}, & 3 - D \\ & \frac{D}{2} \end{array} \middle| \frac{1}{y_1} \right) , \\ \Phi^{(4)}_{[51]}(y_1) &= (y_1)^{2D-5} {}_2F_1 \left(\begin{array}{cc} 5 - 2D, & 4 - \frac{3D}{2} \\ & 2 - \frac{D}{2} \end{array} \middle| \frac{1}{y_1} \right) , \end{aligned} \tag{118}$$

which the convergent region is $|y_1| > 1$. Correspondingly the integral in the region $|y_1| > 1$ is a linear combination of two fundamental solutions:

$$\Phi_{51}(y_1) = C^{(3)}_{[51]}\Phi^{(3)}_{[51]}(y_1) + C^{(4)}_{[51]}\Phi^{(4)}_{[51]}(y_1) \; . \tag{119}$$

As $m_1^2 \ll m_5^2$, $m_2 = m_3 = m_4 = 0$,

$$\begin{aligned} I_1 &= \Big(\Lambda^2_{\rm RE}\Big)^{8-2D} \int \frac{d^D\mathbf{q}}{(2\pi)^D} \frac{1}{(q_1^2 - m_1^2)q_2^2q_3^2(q_1 + q_2 + q_3 + q_4)^2(q_3^2 - m_5^2)} \\ &= I_{1,0} + \cdots \; , \end{aligned} \tag{120}$$

where

$$\begin{aligned} I_{1,0} &= \Big(\Lambda^2_{\rm RE}\Big)^{8-2D} \int \frac{d^D\mathbf{q}}{(2\pi)^D} \frac{1}{q_1^2q_2^2q_3^2(q_1 + q_2 + q_3 + q_4)^2(q_3^2 - m_5^2)} \\ &= \frac{-m_5^6}{(4\pi)^8}\Big(\frac{4\pi\Lambda^2_{\rm RE}}{m_5^2}\Big)^{8-2D}\Gamma(4 - \frac{3D}{2})\Gamma(5 - 2D) \; . \end{aligned} \tag{121}$$

This result indicates

$$C^{(1)}_{[51]} = \frac{-m_5^6}{(4\pi)^8}\Big(\frac{4\pi\Lambda^2_{\rm RE}}{m_5^2}\Big)^{8-2D}\Gamma(4 - \frac{3D}{2})\Gamma(5 - 2D) \; . \tag{122}$$

As $m_1^2 \gg m_5^2$ and $m_2 = m_3 = m_4 = 0$, $I_1 = I_{1,\infty} + \cdots$, where

$$\begin{aligned} I_{1,\infty} &= \Big(\Lambda^2_{\rm RE}\Big)^{8-2D} \int \frac{d^D\mathbf{q}}{(2\pi)^D} \frac{1}{(q_1^2 - m_1^2)q_2^2q_3^2(q_1 + q_2 + q_3 + q_4)^2q_3^2} \\ &= \frac{-m_1^6}{(4\pi)^8}\Big(\frac{4\pi\Lambda^2_{\rm RE}}{m_1^2}\Big)^{8-2D}\Gamma(4 - \frac{3D}{2})\Gamma(5 - 2D) \; . \end{aligned} \tag{123}$$

This result indicates

$$C^{(4)}_{[51]} = \frac{-m_5^6}{(4\pi)^8}\Big(\frac{4\pi\Lambda^2_{\rm RE}}{m_5^2}\Big)^{8-2D}\Gamma(4 - \frac{3D}{2})\Gamma(5 - 2D) \; . \tag{124}$$

Actually, the Mellin-Barnes representation of the Feynman integral in this case can be obtained as

$$\begin{aligned} U_5 = &\frac{-m_5^6}{(4\pi)^8}\Big(\frac{4\pi\Lambda^2_{\rm RE}}{m_5^2}\Big)^{8-2D}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} ds_1\Big(\frac{m_1^2}{m_5^2}\Big)^{s_1}\Gamma(-s_1) \\ &\times\Gamma(\frac{D}{2} - 1 - s_1)\Gamma(4 - \frac{3D}{2} + s_1)\Gamma(5 - 2D + s_1) \; . \end{aligned} \tag{125}$$

The residue of simple pole of $\Gamma(-s_1)$ provides $C^{(1)}_{[51]}\Phi^{(1)}_{[51]}(y_1)$, that of simple pole of $\Gamma(D/2 - 1 - s_1)$ provides $C^{(2)}_{[51]}\Phi^{(2)}_{[51]}(y_1)$, that of simple pole of $\Gamma(4 - \frac{3D}{2} + s_1)$ provides $C^{(3)}_{[51]}\Phi^{(3)}_{[51]}(y_1)$, and

that of simple pole of $\Gamma(5-2D+s_{_1})$ provides $C^{(4)}_{_{[51]}}\Phi^{(4)}_{_{[51]}}(y_{_1})$, respectively. And the residue of the simple pole of $\Gamma(D/2-1-s_{_1})$ and $\Gamma(4-\frac{3D}{2}+s_{_1})$ can induce

$$C^{(2)}_{_{[51]}}=C^{(3)}_{_{[51]}}=\frac{-m^6_{_5}}{(4\pi)^8}\Big(\frac{4\pi\Lambda^2_{_{\rm RE}}}{m^2_{_5}}\Big)^{8-2D}\Gamma(4-\frac{3D}{2})\Gamma(3-D)\Gamma(1-\frac{D}{2})\;. \tag{126}$$

A conclusion for the other two nonzero virtual mass for the four-loop vacuum integral with five propagates, such as $m_{_2}\neq 0$, $m_{_5}\neq 0$, $m_{_1}=m_{_3}=m_{_4}=0$ is analogous to that of $m_{_1}\neq 0$, $m_{_5}\neq 0$, $m_{_2}=m_{_3}=m_{_4}=0$.

C. The four-loop vacuum integrals with six or seven propagates

In our previous works [85, 86], we obtain GKZ hypergeometric systems of some one-loop and two-loop Feynman integrals, which show the algorithm and the obvious hypergeometric series solutions for the one-loop and two-loop Feynman integrals. Recently, the authors of the Refs. [72, 103] also give publicly available computer packages MBConicHulls [72] and FeynGKZ [103] to compute Feynman integrals in terms of hypergeometric functions, which are meaningful to improve computing efficiency. Through the package FeynGKZ [103], they give the examples of some one-loop and two-loop Feynman integrals, which are tested analytically by our previous work [85, 86], as well as numerically using the package FIESTA [109].

Here, we also evaluate the four-loop vacuum integrals with six or seven propagates using FeynGKZ [103], which can be seen in the supplementary material. Note that FeynGKZ can't evaluate the four-loop vacuum integrals with five propagates through Mellin-Barnes representations and Miller's transformation. In the supplementary material, we can see that the GKZ hypergeometric systems of four-loop vacuum integrals with six or seven propagates are in agree with our above results. Using FeynGKZ, we also give some hypergeometric series solutions for the four-loop vacuum integrals with six or seven propagates. The series solutions from GKZ hypergeometric systems for special case with the two nonzero virtual masses and the three nonzero virtual masses are also showed in the supplementary material, and tested numerically using FIESTA [109]. One can see that the computing time using the hypergeometric series solutions is about $\mathcal{O}(10^{-5})$ times that using FIESTA, which evaluate quickly.

Note that, except above five topologies, the four-loop vacuum integrals still have five topologies, which are one topology with seven propagates, two topologies with eight prop-

agates, and two topologies with nine propagates [44]. Due that they have complex mathematical structure, the five topologies of the four-loop vacuum integrals can't obtain the GKZ hypergeometric systems, through Mellin-Barnes representations and Miller's transformation. In next work, we will embed the general four-loop vacuum Feynman integrals into the subvarieties of Grassmannian manifold [87], to explore more possibilities of the general four-loop vacuum Feynman integrals.

VI. CONCLUSIONS

Using Mellin-Barnes representation and Miller's transformation, we derive GKZ hypergeometric systems of the four-loop vacuum integrals with arbitrary masses. The dimension of the GKZ hypergeometric system equals the number of independent dimensionless ratios among the virtual mass squared. In the neighborhoods of origin including infinity, we can obtain analytical hypergeometric series solutions of the four-loop vacuum integrals through GKZ hypergeometric systems. The linear independent hypergeometric series solutions whose convergent regions have non-empty intersection can constitute a fundamental solution system in a proper subset of the whole parameter space. In certain convergent region, the four-loop vacuum integrals can be formulated as a linear combination of the corresponding fundamental solution system. The combination coefficients are determined by the vacuum integral at some ordinary points or regular singularities, or the Mellin-Barnes representation of the vacuum integral.

Here, we obtain the analytical hypergeometric solutions of the four-loop vacuum integrals in the neighborhoods of origin including infinity, using GKZ hypergeometric systems on general manifold. In order to derive the fundamental solution system in neighborhoods of all possible regular singularities, we can embed the vacuum integrals in corresponding Grassmannian manifold [87] through their parametrization. To efficiently derive the fundamental solution system, next we will embed the four-loop vacuum integrals into the subvarieties of Grassmannian using the α-parametric representation.

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APPENDIX A: THE CALCULATION OF THE INTEGRAL I_q

For the integral I_q, firstly we can integrate out q_1:

$$\begin{aligned} I_{q_1} &\equiv \int \frac{d^D q_1}{(2\pi)^D} \frac{1}{(q_1^2)^{1+s_1}((q_1+q_2+q_3+q_4)^2)^{1+s_4}} \\ &= \frac{\Gamma(2+s_1+s_4)}{\Gamma(1+s_1)\Gamma(1+s_4)} \int_0^1 dx (1-x)^{s_1} x^{s_4} \\ &\quad \times \int \frac{d^D q_1}{(2\pi)^D} \frac{1}{[q_1^2 + x(1-x)(q_2+q_3+q_4)^2]^{2+s_1+s_4}} \,. \end{aligned} \tag{A1}$$

Using the well-known integral

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{[q^2+\triangle]^n} = \frac{i(-)^{D/2}\Gamma(n-\frac{D}{2})}{(4\pi)^{D/2}\Gamma(n)} \Big(\frac{1}{\triangle}\Big)^{n-D/2} , \tag{A2}$$

one can have

$$\begin{aligned} I_{q_1} &= \frac{i(-)^{D/2}\Gamma(2-\frac{D}{2}+s_1+s_4)}{(4\pi)^{D/2}\Gamma(1+s_1)\Gamma(1+s_4)} \frac{1}{[(q_2+q_3+q_4)^2]^{2-\frac{D}{2}+s_1+s_4}} \\ &\quad \times \int_0^1 dx x^{D/2-2-s_1}(1-x)^{D/2-2-s_4} \,. \end{aligned} \tag{A3}$$

Through Beta function

$$B(m,n) = \int_0^1 dx\, x^{m-1}(1-x)^{n-1} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \,, \tag{A4}$$

we can have

$$\begin{aligned} I_{q_1} &= \frac{i(-)^{D/2}\Gamma(2-\frac{D}{2}+s_1+s_4)\Gamma(\frac{D}{2}-1-s_1)\Gamma(\frac{D}{2}-1-s_4)}{(4\pi)^{D/2}\Gamma(1+s_1)\Gamma(1+s_4)\Gamma(D-2-s_1-s_4)} \\ &\quad \times \frac{1}{[(q_2+q_3+q_4)^2]^{2-\frac{D}{2}+s_1+s_4}} \,. \end{aligned} \tag{A5}$$

Similarly, we can integrate out q_2:

$$\begin{aligned}
I_{q_2} &\equiv \int \frac{d^D q_2}{(2\pi)^D} \frac{1}{(q_2^2)^{1+s_2}[(q_2+q_3+q_4)^2]^{2-\frac{D}{2}+s_1+s_4}} \\
&= \frac{i(-)^{D/2}\Gamma(3-D+s_1+s_2+s_4)\Gamma(\frac{D}{2}-1-s_2)\Gamma(D-2-s_1-s_4)}{(4\pi)^{D/2}\Gamma(1+s_2)\Gamma(2-\frac{D}{2}+s_1+s_4)\Gamma(\frac{3D}{2}-3-s_1-s_2-s_4)} \\
&\quad \times \frac{1}{[(q_3+q_4)^2]^{3-D+s_1+s_2+s_4}} \;.
\end{aligned} \tag{A6}$$

And then, we integrate out q_3:

$$\begin{aligned}
I_{q_3} &\equiv \int \frac{d^D q_3}{(2\pi)^D} \frac{1}{(q_3^2)^{1+s_3}[(q_3+q_4)^2]^{3-D+s_1+s_2+s_4}} \\
&= \frac{i(-)^{D/2}\Gamma(4-\frac{3D}{2}+\sum_{i=1}^{4}s_i)\Gamma(\frac{D}{2}-1-s_3)\Gamma(\frac{3D}{2}-3-s_1-s_2-s_4)}{(4\pi)^{D/2}\Gamma(1+s_3)\Gamma(3-D+s_1+s_2+s_4)\Gamma(2D-4-\sum_{i=1}^{4}s_i)} \\
&\quad \times \frac{1}{(q_4^2)^{4-\frac{3D}{2}+\sum_{i=1}^{4}s_i}} \;.
\end{aligned} \tag{A7}$$

Last, we integrate out q_4:

$$\begin{aligned}
I_{q_4} &\equiv \int \frac{d^D q_4}{(2\pi)^D} \frac{1}{(q_4^2-m_5^2)(q_4^2)^{4-\frac{3D}{2}+\sum_{i=1}^{4}s_i}} \\
&= \frac{i}{(4\pi)^{D/2}}(-)^{5-\frac{3D}{2}+\sum_{i=1}^{4}s_i}\Big(\frac{1}{m_5^2}\Big)^{5-2D+\sum_{i=1}^{4}s_i}\Gamma(5-2D+\sum_{i=1}^{4}s_i)\Gamma(2D-4-\sum_{i=1}^{4}s_i) \;.
\end{aligned} \tag{A8}$$

Together with Eqs. (A5-A8), one can have

$$\begin{aligned}
I_q &= \frac{-1}{(4\pi)^{2D}}(-)^{\sum_{i=1}^{4}s_i}\Big(\frac{1}{m_5^2}\Big)^{5-2D+\sum_{i=1}^{4}s_i}\Big[\prod_{i=1}^{4}\Gamma(\frac{D}{2}-1-s_i)\Gamma(1+s_i)^{-1}\Big] \\
&\quad \times\Gamma(4-\frac{3D}{2}+\sum_{i=1}^{4}s_i)\Gamma(5-2D+\sum_{i=1}^{4}s_i) \;.
\end{aligned} \tag{A9}$$

APPENDIX B: THE HYPERGEOMETRIC SERIES SOLUTIONS OF THE INTEGER LATTICE $\mathbf{B}_{1345}$

According the basis of integer lattice $\mathbf{B}_{1345}$, one can also construct GKZ hypergeometric series solutions below.

- $I_2 = [2,5,\cdots,8,10]$, i.e. the implement $J_2 = [1,\cdots,10] \setminus I_2 = [1,3,4,9]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(2)}_{[1345]} = y_1^{\frac{D}{2}-1} y_2^{\frac{D}{2}-1} y_4^{\frac{D}{2}-2} \sum_{\mathbf{n}=0}^{\infty} c^{(2)}_{[1345]}(\mathbf{n}) f_{[1345]} \;,$$

$$c^{(2)}_{[1345]}(\mathbf{n}) = \frac{\Gamma(1+\sum\limits_{i=1}^{4} n_i)\Gamma(2-\frac{D}{2}+\sum\limits_{i=1}^{4} n_i)}{\Big[\prod\limits_{i=1}^{4} n_i!\Big]\Gamma(\frac{D}{2}+n_1)\Gamma(\frac{D}{2}+n_2)\Gamma(\frac{D}{2}+n_3)\Gamma(2-\frac{D}{2}+n_4)} \;. \tag{B1}$$

- $I_3 = [2,4,6,7,9,10]$, i.e. the implement $J_3 = [1,\cdots,10] \setminus I_3 = [1,3,5,8]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(3)}_{[1345]} = y_1^{\frac{D}{2}-1} y_3^{\frac{D}{2}-1} y_4^{\frac{D}{2}-2} \sum_{\mathbf{n}=0}^{\infty} c^{(3)}_{[1345]}(\mathbf{n}) f_{[1345]} \;,$$

$$c^{(3)}_{[1345]}(\mathbf{n}) = \frac{\Gamma(1+\sum\limits_{i=1}^{4} n_i)\Gamma(2-\frac{D}{2}+\sum\limits_{i=1}^{4} n_i)}{\Big[\prod\limits_{i=1}^{4} n_i!\Big]\Gamma(\frac{D}{2}+n_1)\Gamma(\frac{D}{2}+n_2)\Gamma(2-\frac{D}{2}+n_3)\Gamma(\frac{D}{2}+n_4)} \;. \tag{B2}$$

- $I_4 = [2,4,\cdots,7,10]$, i.e. the implement $J_4 = [1,\cdots,10] \setminus I_4 = [1,3,8,9]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(4)}_{[1345]} = y_1^{\frac{D}{2}-1} y_4^{D-3} \sum_{\mathbf{n}=0}^{\infty} c^{(4)}_{[1345]}(\mathbf{n}) f_{[1345]} \;,$$

$$c^{(4)}_{[1345]}(\mathbf{n}) = \frac{\Gamma(2-\frac{D}{2}+\sum\limits_{i=1}^{4} n_i)\Gamma(3-D+\sum\limits_{i=1}^{4} n_i)}{\Big[\prod\limits_{i=1}^{4} n_i!\Big]\Gamma(\frac{D}{2}+n_1)\Gamma(\frac{D}{2}+n_2)\Gamma(2-\frac{D}{2}+n_3)\Gamma(2-\frac{D}{2}+n_4)} \;. \tag{B3}$$

- $I_5 = [2,3,6,8,9,10]$, i.e. the implement $J_5 = [1,\cdots,10] \setminus I_5 = [1,4,5,7]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(5)}_{[1345]} = y_2^{\frac{D}{2}-1} y_3^{\frac{D}{2}-1} y_4^{\frac{D}{2}-2} \sum_{\mathbf{n}=0}^{\infty} c^{(5)}_{[1345]}(\mathbf{n}) f_{[1345]} \;,$$

$$c^{(5)}_{[1345]}(\mathbf{n}) = \frac{\Gamma(1+\sum\limits_{i=1}^{4} n_i)\Gamma(2-\frac{D}{2}+\sum\limits_{i=1}^{4} n_i)}{\Big[\prod\limits_{i=1}^{4} n_i!\Big]\Gamma(\frac{D}{2}+n_1)\Gamma(2-\frac{D}{2}+n_2)\Gamma(\frac{D}{2}+n_3)\Gamma(\frac{D}{2}+n_4)} \;. \tag{B4}$$

- $I_6 = [2,3,5,6,8,10]$, i.e. the implement $J_6 = [1,\cdots,10] \setminus I_6 = [1,4,7,9]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(6)}_{[1345]} = y_2^{\frac{D}{2}-1} y_4^{D-3} \sum_{\mathbf{n}=0}^{\infty} c^{(6)}_{[1345]}(\mathbf{n}) f_{[1345]} \;,$$

$$c^{(6)}_{[1345]}(\mathbf{n})=\frac{\Gamma(2-\frac{D}{2}+\sum\limits_{i=1}^{4}n_i)\Gamma(3-D+\sum\limits_{i=1}^{4}n_i)}{\Big[\prod\limits_{i=1}^{4}n_i!\Big]\Gamma(\frac{D}{2}+n_1)\Gamma(2-\frac{D}{2}+n_2)\Gamma(\frac{D}{2}+n_3)\Gamma(2-\frac{D}{2}+n_4)}\;. \tag{B5}$$

- $I_7=[2,3,4,6,9,10]$, i.e. the implement $J_7=[1,\cdots,10]\setminus I_7=[1,5,7,8]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(7)}_{[1345]}=y_3^{\frac{D}{2}-1}y_4^{D-3}\sum_{\mathbf{n}=0}^{\infty}c^{(7)}_{[1345]}(\mathbf{n})f_{[1345]}\;,$$

$$c^{(7)}_{[1345]}(\mathbf{n})=\frac{\Gamma(2-\frac{D}{2}+\sum\limits_{i=1}^{4}n_i)\Gamma(3-D+\sum\limits_{i=1}^{4}n_i)}{\Big[\prod\limits_{i=1}^{4}n_i!\Big]\Gamma(\frac{D}{2}+n_1)\Gamma(2-\frac{D}{2}+n_2)\Gamma(2-\frac{D}{2}+n_3)\Gamma(\frac{D}{2}+n_4)}\;. \tag{B6}$$

- $I_8=[2,3,4,5,6,10]$, i.e. the implement $J_8=[1,\cdots,10]\setminus I_8=[1,7,8,9]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(8)}_{[1345]}=y_4^{\frac{3D}{2}-4}\sum_{\mathbf{n}=0}^{\infty}c^{(8)}_{[1345]}(\mathbf{n})f_{[1345]}\;,$$

$$c^{(8)}_{[1345]}(\mathbf{n})=\frac{\Gamma(3-D+\sum\limits_{i=1}^{4}n_i)\Gamma(4-\frac{3D}{2}+\sum\limits_{i=1}^{4}n_i)}{\Big[\prod\limits_{i=1}^{4}n_i!\Big]\Gamma(\frac{D}{2}+n_1)\Gamma(2-\frac{D}{2}+n_2)\Gamma(2-\frac{D}{2}+n_3)\Gamma(2-\frac{D}{2}+n_4)}\;. \tag{B7}$$

- $I_9=[1,6,\cdots,10]$, i.e. the implement $J_9=[1,\cdots,10]\setminus I_9=[2,3,4,5]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(9)}_{[1345]}=y_1^{\frac{D}{2}-1}y_2^{\frac{D}{2}-1}y_3^{\frac{D}{2}-1}y_4^{\frac{D}{2}-2}\sum_{\mathbf{n}=0}^{\infty}c^{(9)}_{[1345]}(\mathbf{n})f_{[1345]}\;,$$

$$c^{(9)}_{[1345]}(\mathbf{n})=\frac{\Gamma(1+\sum\limits_{i=1}^{4}n_i)\Gamma(2-\frac{D}{2}+\sum\limits_{i=1}^{4}n_i)}{\Big[\prod\limits_{i=1}^{4}n_i!\Big]\Gamma(2-\frac{D}{2}+n_1)\Gamma(\frac{D}{2}+n_2)\Gamma(\frac{D}{2}+n_3)\Gamma(\frac{D}{2}+n_4)}\;. \tag{B8}$$

- $I_{10}=[1,5,\cdots,8,10]$, i.e. the implement $J_{10}=[1,\cdots,10]\setminus I_{10}=[2,3,4,9]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(10)}_{[1345]}=y_1^{\frac{D}{2}-1}y_2^{\frac{D}{2}-1}y_4^{D-3}\sum_{\mathbf{n}=0}^{\infty}c^{(10)}_{[1345]}(\mathbf{n})f_{[1345]}\;,$$

$$c^{(10)}_{[1345]}(\mathbf{n})=\frac{\Gamma(2-\frac{D}{2}+\sum\limits_{i=1}^{4}n_i)\Gamma(3-D+\sum\limits_{i=1}^{4}n_i)}{\Big[\prod\limits_{i=1}^{4}n_i!\Big]\Gamma(2-\frac{D}{2}+n_1)\Gamma(\frac{D}{2}+n_2)\Gamma(\frac{D}{2}+n_3)\Gamma(2-\frac{D}{2}+n_4)}\;. \tag{B9}$$

- $I_{11} = [1,4,6,7,9,10]$, i.e. the implement $J_{11} = [1,\cdots,10] \setminus I_{11} = [2,3,5,8]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(11)}_{[1345]} = y_1^{\frac{D}{2}-1} y_3^{\frac{D}{2}-1} y_4^{D-3} \sum_{\mathbf{n}=0}^{\infty} c^{(11)}_{[1345]}(\mathbf{n}) f_{[1345]} \;,$$

$$c^{(11)}_{[1345]}(\mathbf{n}) = \frac{\Gamma(2-\frac{D}{2}+\sum\limits_{i=1}^{4} n_i)\Gamma(3-D+\sum\limits_{i=1}^{4} n_i)}{\Big[\prod\limits_{i=1}^{4} n_i!\Big]\Gamma(2-\frac{D}{2}+n_1)\Gamma(\frac{D}{2}+n_2)\Gamma(2-\frac{D}{2}+n_3)\Gamma(\frac{D}{2}+n_4)} \;. \tag{B10}$$

- $I_{12} = [1,4,\cdots,7,10]$, i.e. the implement $J_{12} = [1,\cdots,10] \setminus I_{12} = [2,3,8,9]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(12)}_{[1345]} = y_1^{\frac{D}{2}-1} y_4^{\frac{3D}{2}-4} \sum_{\mathbf{n}=0}^{\infty} c^{(12)}_{[1345]}(\mathbf{n}) f_{[1345]} \;,$$

$$c^{(12)}_{[1345]}(\mathbf{n}) = \frac{\Gamma(3-D+\sum\limits_{i=1}^{4} n_i)\Gamma(4-\frac{3D}{2}+\sum\limits_{i=1}^{4} n_i)}{\Big[\prod\limits_{i=1}^{4} n_i!\Big]\Gamma(2-\frac{D}{2}+n_1)\Gamma(\frac{D}{2}+n_2)\Gamma(2-\frac{D}{2}+n_3)\Gamma(2-\frac{D}{2}+n_4)} \;. \tag{B11}$$

- $I_{13} = [1,3,6,8,9,10]$, i.e. the implement $J_{13} = [1,\cdots,10] \setminus I_{13} = [2,4,5,7]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(13)}_{[1345]} = y_2^{\frac{D}{2}-1} y_3^{\frac{D}{2}-1} y_4^{D-3} \sum_{\mathbf{n}=0}^{\infty} c^{(13)}_{[1345]}(\mathbf{n}) f_{[1345]} \;,$$

$$c^{(13)}_{[1345]}(\mathbf{n}) = \frac{\Gamma(2-\frac{D}{2}+\sum\limits_{i=1}^{4} n_i)\Gamma(3-D+\sum\limits_{i=1}^{4} n_i)}{\Big[\prod\limits_{i=1}^{4} n_i!\Big]\Gamma(2-\frac{D}{2}+n_1)\Gamma(2-\frac{D}{2}+n_2)\Gamma(\frac{D}{2}+n_3)\Gamma(\frac{D}{2}+n_4)} \;. \tag{B12}$$

- $I_{14} = [1,3,5,6,8,10]$, i.e. the implement $J_{14} = [1,\cdots,10] \setminus I_{14} = [2,4,7,9]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(14)}_{[1345]} = y_2^{\frac{D}{2}-1} y_4^{\frac{3D}{2}-4} \sum_{\mathbf{n}=0}^{\infty} c^{(14)}_{[1345]}(\mathbf{n}) f_{[1345]} \;,$$

$$c^{(14)}_{[1345]}(\mathbf{n}) = \frac{\Gamma(3-D+\sum\limits_{i=1}^{4} n_i)\Gamma(4-\frac{3D}{2}+\sum\limits_{i=1}^{4} n_i)}{\Big[\prod\limits_{i=1}^{4} n_i!\Big]\Gamma(2-\frac{D}{2}+n_1)\Gamma(2-\frac{D}{2}+n_2)\Gamma(\frac{D}{2}+n_3)\Gamma(2-\frac{D}{2}+n_4)} \;. \tag{B13}$$

- $I_{15} = [1,3,4,6,9,10]$, i.e. the implement $J_{15} = [1,\cdots,10] \setminus I_{15} = [2,5,7,8]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(15)}_{[1345]} = y_3^{\frac{D}{2}-1} y_4^{\frac{3D}{2}-4} \sum_{\mathbf{n}=0}^{\infty} c^{(15)}_{[1345]}(\mathbf{n}) f_{[1345]} \;,$$

$$c^{(15)}_{[1345]}(\mathbf{n}) = \frac{\Gamma(3-D+\sum\limits_{i=1}^{4} n_i)\Gamma(4-\frac{3D}{2}+\sum\limits_{i=1}^{4} n_i)}{\Big[\prod\limits_{i=1}^{4} n_i!\Big]\Gamma(2-\frac{D}{2}+n_1)\Gamma(2-\frac{D}{2}+n_2)\Gamma(2-\frac{D}{2}+n_3)\Gamma(\frac{D}{2}+n_4)} \;. \tag{B14}$$

- $I_{16} = [1,3,4,5,6,10]$, i.e. the implement $J_{16} = [1,\cdots,10] \setminus I_{16} = [2,7,8,9]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(16)}_{[1345]} = y_4^{2D-5} \sum_{\mathbf{n}=0}^{\infty} c^{(16)}_{[1345]}(\mathbf{n}) f_{[1345]} \;,$$
$$c^{(16)}_{[1345]}(\mathbf{n}) = \frac{\Gamma(4-\frac{3D}{2}+\sum\limits_{i=1}^{4} n_i)\Gamma(5-2D+\sum\limits_{i=1}^{4} n_i)}{\prod\limits_{i=1}^{4} n_i!\Gamma(2-\frac{D}{2}+n_i)} \;. \tag{B15}$$

APPENDIX C: THE HYPERGEOMETRIC SERIES SOLUTIONS OF THE INTEGER LATTICE $\mathbf{B}_{\tilde{1}345}$

According the basis of integer lattice $\mathbf{B}_{\tilde{1}345}$, one can construct GKZ hypergeometric series solutions below.

- $I_1 = [1,2,7,\cdots,10]$, i.e. the implement $J_1 = [1,\cdots,10] \setminus I_1 = [3,4,5,6]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(1)}_{[\tilde{1}345]} = y_1^{\frac{D}{2}-1} y_2^{\frac{D}{2}-1} y_3^{\frac{D}{2}-1} y_4^{\frac{D}{2}-1} \sum_{\mathbf{n}=0}^{\infty} c^{(1)}_{[\tilde{1}345]}(\mathbf{n}) f_{[\tilde{1}345]} \;,$$
$$f_{[\tilde{1}345]} = \Big(y_4\Big)^{n_1}\Big(y_1\Big)^{n_2}\Big(y_2\Big)^{n_3}\Big(y_3\Big)^{n_4} \;, \tag{C1}$$

where the coefficient is

$$c^{(1)}_{[\tilde{1}345]}(\mathbf{n}) = \frac{\Gamma(\frac{D}{2}+\sum\limits_{i=1}^{4} n_i)\Gamma(1+\sum\limits_{i=1}^{4} n_i)}{\prod\limits_{i=1}^{4} n_i!\Gamma(\frac{D}{2}+n_i)} \;. \tag{C2}$$

- $I_2 = [1,2,6,\cdots,9]$, i.e. the implement $J_2 = [1,\cdots,10] \setminus I_2 = [3,4,5,10]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(2)}_{[\tilde{1}345]} = y_1^{\frac{D}{2}-1} y_2^{\frac{D}{2}-1} y_3^{\frac{D}{2}-1} \sum_{\mathbf{n}=0}^{\infty} c^{(2)}_{[\tilde{1}345]}(\mathbf{n}) f_{[\tilde{1}345]} \;,$$
$$c^{(2)}_{[\tilde{1}345]}(\mathbf{n}) = \frac{\Gamma(1+\sum\limits_{i=1}^{4} n_i)\Gamma(2-\frac{D}{2}+\sum\limits_{i=1}^{4} n_i)}{\Big[\prod\limits_{i=1}^{4} n_i!\Big]\Gamma(2-\frac{D}{2}+n_1)\Gamma(\frac{D}{2}+n_2)\Gamma(\frac{D}{2}+n_3)\Gamma(\frac{D}{2}+n_4)} \;. \tag{C3}$$

- $I_3=[1,2,5,7,8,10]$, i.e. the implement $J_3=[1,\cdots,10]\setminus I_3=[3,4,6,9]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(3)}_{[\tilde{1}345]}=y_1^{\frac{D}{2}-1}y_2^{\frac{D}{2}-1}y_4^{\frac{D}{2}-1}\sum_{\mathbf{n}=0}^{\infty}c^{(3)}_{[\tilde{1}345]}(\mathbf{n})f_{[\tilde{1}345]}\;,$$
$$c^{(3)}_{[\tilde{1}345]}(\mathbf{n})=\frac{\Gamma(1+\sum\limits_{i=1}^{4}n_i)\Gamma(2-\frac{D}{2}+\sum\limits_{i=1}^{4}n_i)}{\Big[\prod\limits_{i=1}^{4}n_i!\Big]\Gamma(\frac{D}{2}+n_1)\Gamma(\frac{D}{2}+n_2)\Gamma(\frac{D}{2}+n_3)\Gamma(2-\frac{D}{2}+n_4)}\;. \tag{C4}$$

- $I_4=[1,2,5,\cdots,8]$, i.e. the implement $J_4=[1,\cdots,10]\setminus I_4=[3,4,9,10]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(4)}_{[\tilde{1}345]}=y_1^{\frac{D}{2}-1}y_2^{\frac{D}{2}-1}\sum_{\mathbf{n}=0}^{\infty}c^{(4)}_{[\tilde{1}345]}(\mathbf{n})f_{[\tilde{1}345]}\;,$$
$$c^{(4)}_{[\tilde{1}345]}(\mathbf{n})=\frac{\Gamma(2-\frac{D}{2}+\sum\limits_{i=1}^{4}n_i)\Gamma(3-D+\sum\limits_{i=1}^{4}n_i)}{\Big[\prod\limits_{i=1}^{4}n_i!\Big]\Gamma(2-\frac{D}{2}+n_1)\Gamma(\frac{D}{2}+n_2)\Gamma(\frac{D}{2}+n_3)\Gamma(2-\frac{D}{2}+n_4)}\;. \tag{C5}$$

- $I_5=[1,2,4,7,9,10]$, i.e. the implement $J_5=[1,\cdots,10]\setminus I_5=[3,5,6,8]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(5)}_{[\tilde{1}345]}=y_1^{\frac{D}{2}-1}y_3^{\frac{D}{2}-1}y_4^{\frac{D}{2}-1}\sum_{\mathbf{n}=0}^{\infty}c^{(5)}_{[\tilde{1}345]}(\mathbf{n})f_{[\tilde{1}345]}\;,$$
$$c^{(5)}_{[\tilde{1}345]}(\mathbf{n})=\frac{\Gamma(1+\sum\limits_{i=1}^{4}n_i)\Gamma(2-\frac{D}{2}+\sum\limits_{i=1}^{4}n_i)}{\Big[\prod\limits_{i=1}^{4}n_i!\Big]\Gamma(\frac{D}{2}+n_1)\Gamma(\frac{D}{2}+n_2)\Gamma(2-\frac{D}{2}+n_3)\Gamma(\frac{D}{2}+n_4)}\;. \tag{C6}$$

- $I_6=[1,2,4,6,7,9]$, i.e. the implement $J_6=[1,\cdots,10]\setminus I_6=[3,5,8,10]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(6)}_{[\tilde{1}345]}=y_1^{\frac{D}{2}-1}y_3^{\frac{D}{2}-1}\sum_{\mathbf{n}=0}^{\infty}c^{(6)}_{[\tilde{1}345]}(\mathbf{n})f_{[\tilde{1}345]}\;,$$
$$c^{(6)}_{[\tilde{1}345]}(\mathbf{n})=\frac{\Gamma(2-\frac{D}{2}+\sum\limits_{i=1}^{4}n_i)\Gamma(3-D+\sum\limits_{i=1}^{4}n_i)}{\Big[\prod\limits_{i=1}^{4}n_i!\Big]\Gamma(2-\frac{D}{2}+n_1)\Gamma(\frac{D}{2}+n_2)\Gamma(2-\frac{D}{2}+n_3)\Gamma(\frac{D}{2}+n_4)}\;. \tag{C7}$$

- $I_7=[1,2,4,5,7,10]$, i.e. the implement $J_7=[1,\cdots,10]\setminus I_7=[3,6,8,9]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(7)}_{[\tilde{1}345]}=y_1^{\frac{D}{2}-1}y_4^{\frac{D}{2}-1}\sum_{\mathbf{n}=0}^{\infty}c^{(7)}_{[\tilde{1}345]}(\mathbf{n})f_{[\tilde{1}345]}\;,$$

$$c^{(7)}_{[\tilde{1}345]}(\mathbf{n}) = \frac{\Gamma(2-\frac{D}{2}+\sum\limits_{i=1}^{4}n_i)\Gamma(3-D+\sum\limits_{i=1}^{4}n_i)}{\Big[\prod\limits_{i=1}^{4}n_i!\Big]\Gamma(\frac{D}{2}+n_1)\Gamma(\frac{D}{2}+n_2)\Gamma(2-\frac{D}{2}+n_3)\Gamma(2-\frac{D}{2}+n_4)} \,. \tag{C8}$$

- $I_8 = [1,2,4,5,6,7]$, i.e. the implement $J_8 = [1,\cdots,10] \setminus I_8 = [3,8,9,10]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(8)}_{[\tilde{1}345]} = y_1^{\frac{D}{2}-1}\sum_{\mathbf{n}=0}^{\infty} c^{(8)}_{[\tilde{1}345]}(\mathbf{n}) f_{[\tilde{1}345]} \,,$$
$$c^{(8)}_{[\tilde{1}345]}(\mathbf{n}) = \frac{\Gamma(3-D+\sum\limits_{i=1}^{4}n_i)\Gamma(4-\frac{3D}{2}+\sum\limits_{i=1}^{4}n_i)}{\Big[\prod\limits_{i=1}^{4}n_i!\Big]\Gamma(2-\frac{D}{2}+n_1)\Gamma(\frac{D}{2}+n_2)\Gamma(2-\frac{D}{2}+n_3)\Gamma(2-\frac{D}{2}+n_4)} \,. \tag{C9}$$

- $I_9 = [1,2,3,8,9,10]$, i.e. the implement $J_9 = [1,\cdots,10] \setminus I_9 = [4,5,6,7]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(9)}_{[\tilde{1}345]} = y_2^{\frac{D}{2}-1}y_3^{\frac{D}{2}-1}y_4^{\frac{D}{2}-1}\sum_{\mathbf{n}=0}^{\infty} c^{(9)}_{[\tilde{1}345]}(\mathbf{n}) f_{[\tilde{1}345]} \,,$$
$$c^{(9)}_{[\tilde{1}345]}(\mathbf{n}) = \frac{\Gamma(1+\sum\limits_{i=1}^{4}n_i)\Gamma(2-\frac{D}{2}+\sum\limits_{i=1}^{4}n_i)}{\Big[\prod\limits_{i=1}^{4}n_i!\Big]\Gamma(\frac{D}{2}+n_1)\Gamma(2-\frac{D}{2}+n_2)\Gamma(\frac{D}{2}+n_3)\Gamma(\frac{D}{2}+n_4)} \,. \tag{C10}$$

- $I_{10} = [1,2,3,6,8,9]$, i.e. the implement $J_{10} = [1,\cdots,10] \setminus I_{10} = [4,5,7,10]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(10)}_{[\tilde{1}345]} = y_2^{\frac{D}{2}-1}y_3^{\frac{D}{2}-1}\sum_{\mathbf{n}=0}^{\infty} c^{(10)}_{[\tilde{1}345]}(\mathbf{n}) f_{[\tilde{1}345]} \,,$$
$$c^{(10)}_{[\tilde{1}345]}(\mathbf{n}) = \frac{\Gamma(2-\frac{D}{2}+\sum\limits_{i=1}^{4}n_i)\Gamma(3-D+\sum\limits_{i=1}^{4}n_i)}{\Big[\prod\limits_{i=1}^{4}n_i!\Big]\Gamma(2-\frac{D}{2}+n_1)\Gamma(2-\frac{D}{2}+n_2)\Gamma(\frac{D}{2}+n_3)\Gamma(\frac{D}{2}+n_4)} \,. \tag{C11}$$

- $I_{11} = [1,2,3,5,8,10]$, i.e. the implement $J_{11} = [1,\cdots,10] \setminus I_{11} = [4,6,7,9]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(11)}_{[\tilde{1}345]} = y_2^{\frac{D}{2}-1}y_4^{\frac{D}{2}-1}\sum_{\mathbf{n}=0}^{\infty} c^{(11)}_{[\tilde{1}345]}(\mathbf{n}) f_{[\tilde{1}345]} \,,$$
$$c^{(11)}_{[\tilde{1}345]}(\mathbf{n}) = \frac{\Gamma(2-\frac{D}{2}+\sum\limits_{i=1}^{4}n_i)\Gamma(3-D+\sum\limits_{i=1}^{4}n_i)}{\Big[\prod\limits_{i=1}^{4}n_i!\Big]\Gamma(\frac{D}{2}+n_1)\Gamma(2-\frac{D}{2}+n_2)\Gamma(\frac{D}{2}+n_3)\Gamma(2-\frac{D}{2}+n_4)} \,. \tag{C12}$$

- $I_{12} = [1,2,3,5,6,8]$, i.e. the implement $J_{12} = [1,\cdots,10] \setminus I_{12} = [4,7,9,10]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(12)}_{[\tilde{1}345]} = y_2^{\frac{D}{2}-1} \sum_{\mathbf{n}=0}^{\infty} c^{(12)}_{[\tilde{1}345]}(\mathbf{n}) f_{[\tilde{1}345]} \;,$$
$$c^{(12)}_{[\tilde{1}345]}(\mathbf{n}) = \frac{\Gamma(3-D+\sum\limits_{i=1}^{4} n_i)\Gamma(4-\frac{3D}{2}+\sum\limits_{i=1}^{4} n_i)}{\Big[\prod\limits_{i=1}^{4} n_i!\Big]\Gamma(2-\frac{D}{2}+n_1)\Gamma(2-\frac{D}{2}+n_2)\Gamma(\frac{D}{2}+n_3)\Gamma(2-\frac{D}{2}+n_4)} \;. \tag{C13}$$

- $I_{13} = [1,2,3,4,9,10]$, i.e. the implement $J_{13} = [1,\cdots,10] \setminus I_{13} = [5,6,7,8]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(13)}_{[\tilde{1}345]} = y_3^{\frac{D}{2}-1} y_4^{\frac{D}{2}-1} \sum_{\mathbf{n}=0}^{\infty} c^{(13)}_{[\tilde{1}345]}(\mathbf{n}) f_{[\tilde{1}345]} \;,$$
$$c^{(13)}_{[\tilde{1}345]}(\mathbf{n}) = \frac{\Gamma(2-\frac{D}{2}+\sum\limits_{i=1}^{4} n_i)\Gamma(3-D+\sum\limits_{i=1}^{4} n_i)}{\Big[\prod\limits_{i=1}^{4} n_i!\Big]\Gamma(\frac{D}{2}+n_1)\Gamma(2-\frac{D}{2}+n_2)\Gamma(2-\frac{D}{2}+n_3)\Gamma(\frac{D}{2}+n_4)} \;. \tag{C14}$$

- $I_{14} = [1,2,3,4,6,9]$, i.e. the implement $J_{14} = [1,\cdots,10] \setminus I_{14} = [5,7,8,10]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(14)}_{[\tilde{1}345]} = y_3^{\frac{D}{2}-1} \sum_{\mathbf{n}=0}^{\infty} c^{(14)}_{[\tilde{1}345]}(\mathbf{n}) f_{[\tilde{1}345]} \;,$$
$$c^{(14)}_{[\tilde{1}345]}(\mathbf{n}) = \frac{\Gamma(3-D+\sum\limits_{i=1}^{4} n_i)\Gamma(4-\frac{3D}{2}+\sum\limits_{i=1}^{4} n_i)}{\Big[\prod\limits_{i=1}^{4} n_i!\Big]\Gamma(2-\frac{D}{2}+n_1)\Gamma(2-\frac{D}{2}+n_2)\Gamma(2-\frac{D}{2}+n_3)\Gamma(\frac{D}{2}+n_4)} \;. \tag{C15}$$

- $I_{15} = [1,2,3,4,5,10]$, i.e. the implement $J_{15} = [1,\cdots,10] \setminus I_{15} = [6,7,8,9]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(15)}_{[\tilde{1}345]} = y_4^{\frac{D}{2}-1} \sum_{\mathbf{n}=0}^{\infty} c^{(15)}_{[\tilde{1}345]}(\mathbf{n}) f_{[\tilde{1}345]} \;,$$
$$c^{(15)}_{[\tilde{1}345]}(\mathbf{n}) = \frac{\Gamma(3-D+\sum\limits_{i=1}^{4} n_i)\Gamma(4-\frac{3D}{2}+\sum\limits_{i=1}^{4} n_i)}{\Big[\prod\limits_{i=1}^{4} n_i!\Big]\Gamma(\frac{D}{2}+n_1)\Gamma(2-\frac{D}{2}+n_2)\Gamma(2-\frac{D}{2}+n_3)\Gamma(2-\frac{D}{2}+n_4)} \;. \tag{C16}$$

- $I_{16} = [1,2,3,4,5,6]$, i.e. the implement $J_{16} = [1,\cdots,10] \setminus I_{16} = [7,8,9,10]$. The corresponding hypergeometric series solution is written as

$$\Phi^{(16)}_{[\tilde{1}345]} = \sum_{\mathbf{n}=0}^{\infty} c^{(16)}_{[\tilde{1}345]}(\mathbf{n}) f_{[\tilde{1}345]} \;,$$

$$c^{(16)}_{[\tilde{1}345]}(\mathbf{n}) = \frac{\Gamma(4-\frac{3D}{2}+\sum\limits_{i=1}^{4} n_i)\Gamma(5-2D+\sum\limits_{i=1}^{4} n_i)}{\prod\limits_{i=1}^{4} n_i!\Gamma(2-\frac{D}{2}+n_i)} \,. \tag{C17}$$

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